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Doctoral Thesis

Computations on Field Theories with Super- and Higher Spin-Symmetry: Black Holes and Localization

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Abstract

In this thesis we study some aspects of field theories with super and higher spin symmetry. In the context of higher spin symmetry we restrict the study to 3 dimensions, we analyze the phase space of black hole solutions in the setting of $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$ Chern-Simons theory, to then move on to the ∞ -dimensional algebra setting $\mathfrak{hs}(\lambda) \oplus \mathfrak{hs}(\lambda)$, where we construct black hole solutions and we couple these backgrounds to matter.

We finish by introducing the idea of supersymmetric localization and we apply it to a four dimensional $\mathcal{N} = 2$ theories placed on a $SU(2) \times U(1)$ isometric S^4 .

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1. INTRODUCTION

Quantum field theories (QFTs) are likely the most powerful tools theoretical physics has to tackle fundamental problems. The perturbative approach has been successful in giving predictions with astonishing accuracy, and yet we are far from a complete understanding. There is no doubt about the success of the perturbative expansion, but blind amplitude computations even for *a priori* simple Lagrangians could be very hard already at few loops. On the other hand considering systems with more symmetries typically make the Lagrangians more complicated (or even we might not be able to have a Lagrangian for such theory), but it seems that these classes of theories are the ones we understand the better or at least in a more complete picture, so in a sense we can say they are the “simpler” [1].

Therefore in order to understand a little bit deeper field theories, it is worth to study them constrained by this additional symmetries, and in any set up in which we can learn about non-perturbative regime.

The Coleman-Mandula theorem [2] puts strong restrictions on the symmetries the S matrix of an interacting relativistic field theory in flat spacetime can have. This is, we can extend the spacetime symmetries, at most to the conformal group plus internal symmetries which commutes with the Poincare group. As any theorem, the Coleman-Mandula one has its hypotheses, among them we have that the space-time symmetry does not change the statistics of the particles (Fermions transform into fermions and bosons into bosons) and that the theory is formulated in a flat background. Supersymmetry [3] and higher spin (HS) theories [4, 5] are precisely theories which came to life by relaxing these conditions.

The initial motivation of Fradkin and Vasiliev to formulate higher spin theories was to have an alternative candidate to superstring theories in finding and unifying picture of the known interactions and gravity. Today, there are some evidences suggesting that higher spin theories, may arise from the string theories in the tensionless limit. Also, recently proposed dualities with vector models and minimal models [6, 7] in three and two dimensions have turned much more attention on these theories, providing interesting settings to understand more AdS/CFT like dualities [8, 9, 10].

In order to evade the no-go conditions of the Coleman-Mandula theorem, Fradkin and Vasiliev expanded the theory in $(A)dS$ background where there is no notion of S matrix and allowed a non bounded spin spectrum, constructing then consistent interactions originally in four dimensions. Subsequent work of Vasiliev has seen the extension to results in several dimensions and the inclusion of matter [11, 12, 13]. Three dimensions however seems to be special for theories containing gravity, and there are several facts which simplify the discussion of higher spin theories in this

case.

First, the pure gravitational (HS) theory does not propagate bulk degrees of freedom and was thought to be trivial, and a quantum solvable system [14]. Years after though, black hole (BH) solutions were found in [15, 16], they are generally called BTZ black holes, and were used later to study BH microstate counting [17, 18].

Second, in 3D, the pure gravity system can be written using Chern-Simons (CS) gauge theory for a pair of connections [14, 19], with opposite CS levels. The first theory of higher spin fields using CS formulations goes back to [20] and uses an infinite dimensional algebra, and therefore an infinite tower of HS fields.

Third, in 3D, there is no need to include an infinite tower of spins to have a consistent theory [21]. Indeed in [21] higher spin theories were formulated in terms of $\mathfrak{sl}(N, \mathbb{R}) \times \mathfrak{sl}(N, \mathbb{R})$, the theory couples one integer spin field from 2 to N .

By taking the Chern-Simons connections in the $\mathfrak{sl}(3, \mathbb{R})$ algebra, and demanding asymptotically AdS_3 boundary conditions, the asymptotic symmetry algebra (ASA) of conserved charges was found to be $\mathcal{W}_3 \times \mathcal{W}_3$ [21, 22]. These results are claimed to extend for any N , and also for the infinite dimensional Lie algebra in [22] was found the \mathcal{W}_∞ algebra as ASA.

Containing gravity, higher spin theories could have generalizations of the BTZ black hole solutions, in [23, 24] the conditions that the gauge connections should bear to define a black hole solution were proposed. They are based on the following guide: The BH solution should have as smooth limit the BTZ once we are taking the higher spin chemical potentials to zero. It should have the proper smoothness conditions for all the fields at the horizon, and, it needs to fulfill the first law of thermodynamics. These requirements are posed in terms of gauge invariant quantities, in rough words the conditions mimic the BTZ solution, namely, the holonomy of vielbein around the euclidean time cycle evaluated at the horizon is trivial, with this the black hole solution meets the required properties. Using this proposal several $\mathfrak{sl}(N, \mathbb{R})$ BH solutions were constructed for $N = 3, 4$ [23, 24, 25, 26, 27] and [23, 28] for $\mathfrak{hs}(\lambda)$ a ∞ -dimensional algebra. The thermodynamics associated to these classes of solutions were analyzed in several papers [26, 29, 30, 31, 32, 33]. The holonomy condition imposes the proper thermodynamical relations between charges and chemical potentials, and the question now is what is the nature of the charges that are turned on? Namely are they higher spin? To answer this, it is necessary to find the asymptotic symmetry algebra, analyses have been performed in [25, 34, 35]. The analysis for the $\mathfrak{hs}(\lambda)$ algebras are still missing.

Trying to make a comprehensive summary of the developments in supersymmetry is close to pointless, the quantity of related work from the experiment to applications in mathematics is gigantic.

We will rather describe briefly the conceptual ideas about supersymmetric localization and some of the main results that have been obtained using it.

The idea is based on mathematical results [36, 37, 38] and it was used by Witten in several field theory contexts [39, 40, 41].

If we have a supersymmetric Lagrangian theory, with a non-anomalous supersymmetry \mathcal{Q} , we can deform the action by a \mathcal{Q} -exact deformation, and then the

expectation value of \mathcal{Q} closed observables will not change¹. This freedom allows to choose the deformation in a regime in which, to the path integral there will be contributions only from the fixed point set of the chosen deformation, plus, possible contributions of 1-loop fluctuations around this fixed point set.

The simplicity of this result turns, first into a very powerful computational machinery, which simplifies considerably the infinite dimensional path integrals of supersymmetric theories, and second into a very versatile tool which can be applied with diverse purposes. After Witten, Nekrasov used it to compute the integral over the instanton moduli space, deriving “microscopically” the Seiberg-Witten prepotential [42]. Then, Pestun (using also the instanton counting of Nekrasov) used the technique to localize $\mathcal{N} = 2$, $\mathcal{N} = 2^*$ and $\mathcal{N} = 4$ 4D partition functions and Wilson loops on S^4 , proving a conjecture by Erikson-Semenoff-Zarembo and Drukker-Gross, that the Wilson loops in $\mathcal{N} = 4$ SYM are computed by a matrix model. Indeed Pestun showed that the matrix model is gaussian. Using the results of Pestun, Alday-Gaiotto-Tachikawa conjectured that Liouville theory conformal blocks and correlation functions on a Riemann surface of genus g and n punctures are computed by the Nekrasov partition function of $\mathcal{N} = 2$ SCFTs of class S, this is known as AGT duality. Then Kapustin-Willet-Yaakov computed Wilson loops in Chern-Simons theories with matter and showed that the path integrals reduced to non-gaussian matrix models, these results were used later by Drukker-Mariño-Putrov to find the scaling of the planar free energy in ABJM theories², which matches at strong coupling the classical IIA supergravity action on $AdS_4 \times CP_3$ and gives the correct $N^{3/2}$ scaling for the number of degrees of freedom of the M_2 brane theory. After, along the same lines Jafferis compute the sphere partition function, Z of three dimensional theories with four supercharges and R -symmetry, his result shows that the magnitude of that partition function is extremized for the superconformal R -charge of the 3D infrared conformal field theory. This result is known as Z -extremization.

So localization has a very extensive reach, the many different applications passed by, computation of non-local BPS observables, supergravity localization, generalization of index computations, tests of AdS/CFT correspondence, mirror symmetry, and others.

This thesis is organized as follows:

In chapter (2) we make an introduction to Chern-Simons gravity and higher spin theories, we recall the classical results: the BTZ black holes, the Brown-Henneaux central charge and the Virasoro symmetry in the boundary of AdS_3 , we review Regge-Teitelboim treatment of Chern-Simons theories in the presence of boundaries, the appearance of the Kac-Moody algebra and the computation of the asymptotic symmetry algebra of global charges. We present some of the important results in $\mathfrak{sl}(3, \mathbb{R}) \times \mathfrak{sl}(3, \mathbb{R})$ theory, the $\mathcal{W}_3 \times \mathcal{W}_3$ asymptotic symmetry algebra found by Campoleoni *et al.* for connections which asymptote AdS_3 connections. Next we present the conditions proposed by Gutperle-Kraus, to generalize black holes to the case including higher spin symmetry. We present a summary of the black hole solutions

¹ Provided we do not have any boundary contribution in field theory space

² $\mathcal{N} = 6$, $U(N) \times U(N)$ and $SU(N) \times SU(N)$ superconformal CS theories with matter

proposed in the literature and we say few introductory words about $\mathfrak{hs}(\lambda)$ theory. In chapter (3) we used the concepts introduced previously to compute the asymptotic symmetry algebra for $\mathfrak{sl}(3, \mathbb{R}) \times \mathfrak{sl}(3, \mathbb{R})$ black hole solutions. By taking the relevant Kac-Moody algebra and Dirac reduction, we first derive again the result of Campoleoni *et al.*, then we go on to the black hole solutions and imposed the constraints these solutions enforced, at this stage we are not able to find the $\mathcal{W}_3 \times \mathcal{W}_3$ claimed by Compère-Song, for the Gutperle-Kraus black solution. We reproduce their computation which does not use Dirac reduction but rather a full treatment *alla* Regge-Teitelboim and perturbation theory, next we interpret this $\mathcal{W}_3 \times \mathcal{W}_3$ as coming from a non-residual gauge transformation, a transformation which does not respect the boundary conditions. We then embed the Gutperle-Kraus solution in bigger phase space, this phase space will contain less constraints. We use the Dirac reduction, obtaining $\mathcal{W}_3^{(2)} \times \mathcal{W}_3^{(2)}$ algebra.

In chapter (4) we present a family of black hole solutions for higher spin theories with $\mathfrak{hs}(\lambda) \times \mathfrak{hs}(\lambda)$ valued connections, we build them as deformations of BTZ black hole. By construction these solutions satisfy the proper holonomy constraint. We then couple to these backgrounds a pair of scalars through the linearized Vasiliev system, the physical degrees of freedom are contained in master fields taking values in $\mathfrak{hs}(\lambda)^3$, in principle the Vasiliev equations coupled all the components of the master fields. For the black hole solutions presented, though, one is able to find a closed subsystem of equations and finally a solution for the physical degree of freedom. The outcome is differential equations for the physical degree of freedom of order 4,6,8,..., depending on the spin of the deformation. These differential equations are solved in terms of second order differential equations, and solutions behaving as quasinormal modes found, as well as the bulk 2-point function. We end the chapter by making connections with other backgrounds in the literature and with some remarks.

In chapter (5) we give the general idea of supersymmetric localization and some words about finding rigid supersymmetry in curved backgrounds.

In chapter (6) we apply supersymmetric localization to an $SU(2) \times U(1)$ isometric background with the topology of S^4 , We present the generalized Killing spinor equations proposed by Hama-Hosomichi in the context of the 4D ellipsoid. We write then the supersymmetry algebra on vector and matter multiplets as well as the respective actions. We then report the solution for the background fields and spinors, and constraint different functions on this background in the name of regularity. We then performed localization of the path integral, and argue that the fixed point solutions are the same as in S^4 , we then compute the 1-loop determinant of the fluctuations and add also the instantonic piece, the final result happens to be the same as the one in round S^4 . We conclude with some remarks. The appendix (A) is devoted to conventions of the algebras used in the chapters (2), (3), (4), in appendix (B) we have some useful results for computation of the asymptotic symmetry algebra of (3), appendix (C) presents a miscellaneous of results for the black solutions found in (4) as well as the coupling to the scalar field. Appendix (D) is dedicated to conventions

³ Strictly speaking the master field C containing the scalar degree of freedom takes values in $\mathfrak{hs}(\lambda) \cup V_0^1$ with V_0^1 the identity element, actually the physical scalar is precisely the component along the identity.

for the localization computation of chapter (6), and the appendix (E) contains some manipulations for the 1-loop determinant computation.

2. INTRODUCTION TO CHERN-SIMONS THEORIES OF GRAVITY AND HIGHER SPINS

The technical details that allowed Vasiliev [4, 5, 43, 44] to formulate higher spin theories are originally presented using a deformed oscillator algebra, the Moyal product, and the unfolding method; a technique that exhibits the symmetries of the field equations, it is technically involved, and in this thesis we are not going to use it. We will just make connections with the formalism implemented and the original formulation, details of the original treatment can be found in [45, 46, 47, 48]. In what follows we use the setup and results of [21], which allows us to use Chern-Simons theory. This is a very special fact of 3D and it brings down a big deal of simplification.

2.1 3D Gravity

In this section we will review some few facts about 3D gravity, they can be found in [14, 15, 16, 19, 49, 50].

The Hilbert-Einstein action in 3D with negative cosmological constant reads:

$$S_{H-E} = \frac{1}{16\pi G} \int d^3x \sqrt{g} \left(R + \frac{2}{l^2} \right) + B. \quad (2.1)$$

B is a proper boundary term.

In three dimensions, gravity displays rather special facts. A first look the theory seems to be not renormalizable by power counting, but actually any singularity can be absorbed in the renormalization of the cosmological constant [14, 49]. The pure theory does not have gravitational waves, and one may say is trivial, since actually there is no bulk dynamics. The phase space of the theory is trivial (the space of solutions of the classical equations of motion is finite dimensional). And there is also a rather surprising discovery by Brown-Henneaux [51], in which the asymptotic symmetry algebra of global charges was found for asymptotically AdS_3 , and happen to be the two copies of a central extension of the Virasoro algebra, with central charge c given by $c = \frac{3l}{2G}$. The appearance of the Virasoro algebras may look intuitive in a sense, since it can be inferred from the asymptotic of the AdS_3 metric. The central extension instead is less intuitive.

Even though the theory is trivial in the bulk it has black holes, these were found by Bañados, Teitelboim and Zanelli in [15, 16] and generally called BTZ black holes,

we write them here as in [52]:

$$ds^2 = l^2 \left(d\rho^2 + \frac{2G}{l} ((Ml - J)dx_+^2 + (Ml + J)dx_-^2) - (e^{2\rho} + \frac{64\pi^2 G^2}{l^2} (M^2 l^2 - J^2) e^{-2\rho}) dx_+ dx_- \right), \quad (2.2)$$

where $x_{\pm} = t/l \pm \phi$. Indeed the the space of asymptotic AdS_3 solutions is parametrized by the following exact solution of the Einstein's equations [52],

$$ds^2 = l^2 \left(d\rho^2 - \frac{8\pi G}{l} (\mathcal{L}(x_+)dx_+^2 + \bar{\mathcal{L}}(x_-)dx_-^2) - (e^{2\rho} + \frac{64\pi^2 G^2}{l^2} \mathcal{L}\bar{\mathcal{L}}e^{-2\rho}) dx_+ dx_- \right), \quad (2.3)$$

The presence of BHs makes the 3D gravity much more rich, and the central extension of the Virasoro algebra found by Brown and Henneaux comes to play a very important role when trying to understand the thermodynamics of BH solutions [17, 18].

There is another (quite handy) fact about 3D gravity. In [14, 19] was discovered and studied the relation of 3D gravity with Chern-Simons gauge theories. If we write gravity in terms of the frame formalism using the vielbein e_{μ}^a and the spin connection ω_{μ}^{ab} , e and ω are field with indices in the $\mathfrak{so}(2, 1)$ algebra and can be combined in one gauge connection, this is not an unique feature of 3D dimensional theories. What is remarkable about 3D is that we can recast the Hilbert-Einstein action (and the the equations of motion) in terms of a gauge invariant action, namely, the Chern-Simons action:

$$I_k = \kappa \int \text{tr}(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}), \quad (2.4)$$

κ is a coupling constant which depends on global aspects of the gauge group. Following [49], and using the fact that $\omega_{\mu}^{ab} = \epsilon^{ab}_c \omega_{\mu}^c$ we write:

$$\mathcal{A}_{\mu}^a = \omega_{\mu}^a + \frac{e_{\mu}^a}{l} \quad \text{and} \quad \bar{\mathcal{A}}_{\mu}^a = \omega_{\mu}^a - \frac{e_{\mu}^a}{l}, \quad (2.5)$$

where $(\mathcal{A}, \bar{\mathcal{A}})$ are $\mathfrak{so}(2, 1) \oplus \mathfrak{so}(2, 1) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ valued connections. Then, S_{H-E} is given in terms of the CS actions for the pair of connections $(\mathcal{A}, \bar{\mathcal{A}})$ as:

$$S_{H-E} = I_k + I_{-k} + B, \quad (2.6)$$

with $\kappa = \frac{k}{4\pi}$, and k integer. Comparing Chern-Simons and Hilbert-Einstein actions one gets $k = l/(16G)$

For the asymptotic AdS_3 solutions we wrote above, one writes:

$$\begin{aligned} \mathcal{A}_{AdS} &= b^{-1} (L_1 + \frac{2\pi}{k} \mathcal{L}(x_+) L_{-1}) b dx_+ + b^{-1} \partial_{\rho} b d\rho, \\ \bar{\mathcal{A}}_{AdS} &= -b (\frac{2\pi}{k} \bar{\mathcal{L}}(x_-) L_1 + L_{-1}) b^{-1} dx_- + b \partial_{\rho} b^{-1} d\rho \end{aligned} \quad (2.7)$$

with $b(\rho) = e^{\rho L_0}$, and L_{-1}, L_0, L_1 are the three generators of $\mathfrak{sl}(2, \mathbb{R})$, and for both independent connections in (2.7) \mathcal{A} and $\bar{\mathcal{A}}$ we will choose the same set of generators, whose commutation rules are written in the appendix (A).

Now that we have turned the discussion in terms of Chern-Simons theory, let us suppose we have placed the theory on a manifold M with boundary, say $M = \mathbb{R} \times \Sigma$, where Σ is a two dimensional manifold with boundary S_1 . Suppose we have \mathcal{A} valued in a semi-simple Lie algebra \mathfrak{g} , by taking the variation of I_k , and imposing the EOM we will get a boundary piece.

$$\delta I_k = -\frac{k}{4\pi} \int_{\mathbb{R} \times S^1} \text{tr}(\mathcal{A} \wedge \delta \mathcal{A}), \quad (2.8)$$

Therefore in a manifold with boundary we have to impose boundary conditions or to add boundary pieces if we want to be compatible with the action principle, furthermore we need to impose boundary conditions for the fields if we ever expect to quantize the theory, these boundary conditions come to play a fundamental role, they will provide in general an infinite number of global charges satisfying an algebra that depends on the boundary conditions [51]. Different boundary conditions in general lead to different phase spaces of the theory. And even too restrictive boundary conditions may kill all the dynamics.

What is more natural in order to study CS theories then, is to consider some boundary conditions of interest, and to add the proper boundary term if needed in order to make the action compatible with the action principle (differentiable action). In addition to the Dirac formalism, there is another one which is commonly used in these systems to find the phase space and the algebra of the global charges, it was constructed by Regge and Teitelboim [53]. We will now review it.

2.2 Regge-Teitelboim Approach

Let us begin by splitting the $(2,1)$ -connection as:

$$\mathcal{A} = \mathcal{A}_t dt + \mathcal{A}_i dx^i, \quad (2.9)$$

The CS action is then rewritten as :

$$\begin{aligned} I_k &= \frac{k}{4\pi} \int_M dt \wedge dx^i \wedge dx^j \text{tr}(\mathcal{A}_t \mathcal{F}_{ij} - \mathcal{A}_i \dot{\mathcal{A}}_j) + \frac{k}{4\pi} \int_{\partial M} dt \wedge dx^i \text{tr}(\mathcal{A}_t \mathcal{A}_i), \\ &= \frac{k}{4\pi} \int dt \int_{\Sigma} ds \epsilon^{ij} \text{tr}(\mathcal{A}_t \mathcal{F}_{ij} - \mathcal{A}_i \dot{\mathcal{A}}_j) + \frac{k}{4\pi} \int_{\partial M} dt \wedge dx^i \text{tr}(\mathcal{A}_t \mathcal{A}_i). \end{aligned} \quad (2.10)$$

So we have written our action in Hamiltonian form. The action written like this has naively $2 \dim \mathfrak{g}$ dynamical variables encoded in \mathcal{A}_i and $\dim \mathfrak{g}$ Lagrange multipliers contained in \mathcal{A}_t , imposing the constraint $\mathcal{F}_{ij} = 0$ (We still need to impose $\dim \mathfrak{g}$ gauge conditions). The Lagrange multipliers enforce first class constraints ¹ generating

¹ Look [54] for the treatment of gauge theories with constraints

gauge transformations.

We can also read off the equal time Poisson bracket of the canonical variables:

$$\{\mathcal{A}_i^a, \mathcal{A}_j^b\}_{PB} = -\frac{2\pi}{k} g^{ab} \epsilon_{ij} \delta^2 \quad (2.11)$$

where we made explicit the gauge algebra index by writing $\mathcal{A} = \mathcal{A}^a t_a$, the g^{ab} is the inverse of the killing metric $\text{tr}(t_a t_b)$ and δ^2 is the Dirac delta function. Then the Poisson bracket applied to any two functionals of the canonical variables is given by:

$$\{\mathcal{F}, \mathcal{G}\}_{PB} = -\frac{2\pi}{k} \int_{\Sigma} dx^i \wedge dx^j \text{tr} \left(\frac{\delta \mathcal{F}}{\delta \mathcal{A}_i^a} \frac{\delta \mathcal{G}}{\delta \mathcal{A}_j^b} \right) = -\frac{2\pi}{k} \int_{\Sigma} ds \frac{\delta \mathcal{F}}{\delta \mathcal{A}_i^a} \epsilon_{ij} g^{ab} \frac{\delta \mathcal{G}}{\delta \mathcal{A}_j^b} \quad (2.12)$$

We can define the smeared generators of the gauge transformation:

$$G(\Lambda) = \frac{k}{4\pi} \int_{\Sigma} dx^i \wedge dx^j \text{tr}(\Lambda \mathcal{F}_{ij}) + Q(\Lambda), \quad (2.13)$$

where $Q(\Lambda)$ will be a boundary term and will be precisely the global charge. If one considers that Λ is independent of the fields one can write:

$$Q(\Lambda) = \frac{-k}{2\pi} \int_{\partial\Sigma} dx^i \text{tr}(\Lambda \mathcal{A}_i), \quad (2.14)$$

Then the Poisson bracket of two smeared generators is:

$$\{G(\Lambda_1), G(\Lambda_2)\}_{PB} = G([\Lambda_1, \Lambda_2]) + \frac{k}{2\pi} \int_{\partial\Sigma} dx^i \text{tr}(\Lambda_1 \partial_i \Lambda_2) \quad (2.15)$$

Proper gauge transformations are those for which the surface term vanishes. Namely

$$\delta \mathcal{A}_i^a = D_i \Lambda^a = \{\mathcal{A}_i^a, G^a(\Lambda)\}_{PB} \text{ if } \Lambda|_{\partial\Sigma} = 0 \quad (2.16)$$

The transformations such that $Q(\Lambda)$ is different from zero are no honest gauge transformation. They are global transformations mapping physically nonequivalent solutions. After gauge fixing and solving the constraints the $Q(\Lambda)$'s define global charges of the CS theory, inheriting the same algebra as the smeared generators.

$$\{Q(\Lambda_1), Q(\Lambda_2)\}_{PB} = Q([\Lambda_1, \Lambda_2]) + \frac{k}{2\pi} \int_{\partial\Sigma} dx^i \text{tr}(\Lambda_1 \partial_i \Lambda_2) \quad (2.17)$$

Let us write now our expressions using the following $(2, 1)$ -coordinates the (ρ, ϕ, t) one writes $x_{\pm} = t \pm \phi$ and then (2.8) reads:

$$\delta I_k = -\frac{k}{4\pi} \int_{\mathbb{R} \times S^1} \text{tr}(\mathcal{A}_+ \delta \mathcal{A}_- - \mathcal{A}_- \delta \mathcal{A}_+) \quad (2.18)$$

So we can choose $\mathcal{A}_- = 0$ at the boundary, this is typically used in the case $sl(2, \mathbb{R})$ connections, *i.e* gravity [21]. We fixed the gauge by

$$\mathcal{A}_{\rho} = b^{-1}(\rho) \partial_{\rho} b(\rho) \quad (2.19)$$

The constraints $\mathcal{F}_{ij} = \text{impose}$

$$\mathcal{A}_\phi = b^{-1}(\rho)A_\phi(t, \phi)b(\rho), \quad (2.20)$$

and the gauge parameters that respects the boundary condition, will be of the form:

$$\Lambda(\rho, \phi, t) = b^{-1}(\rho)\lambda(\phi, t)b(\rho), \quad (2.21)$$

so,

$$Q(\Lambda) = -\frac{k}{2\pi} \int_{S_1} \text{tr}(\lambda(\phi)A_\phi(\phi)), \quad (2.22)$$

²where λ is the gauge parameter preserving the gauge choice. Replacing this and the constraints in the algebra for a parameter $\lambda^a = \delta(\phi)$ we get the the following affine(Kac-Moody) algebra:

$$\{A^a(\phi), A^b(\phi')\} = -\frac{2\pi}{k}(\delta(\phi - \phi')f^{ab}{}_c A^c - \delta'(\phi - \phi')g^{ab}). \quad (2.23)$$

where g^{ab} is the inverse of the Killing metric, $g_{ab} = \text{tr}(t_a t_b)$, and $f^{ab}{}_c$ are the structure constants of the Lie algebra and $[t_a, t_b] = f_{ab}{}^c t_c$. Here we have used g and g^{-1} to rise and lower the indices. Upon imposing on (2.23), the data of the $\mathfrak{sl}(2, \mathbb{R})$ algebra and the constraint condition coming from the asymptotically AdS_3 solution, *i.e* $A_\phi^1 = 1$ (this is a first class constraint allows us to fix $A_\phi^0 = 0$, we have fix the gauge completely) we get the Brown-Henneaux result.

$$i\{\mathcal{L}_m, \mathcal{L}_n\} = (m - n)\mathcal{L}_{n+m} + \frac{k}{2}n^3\delta_{n,-m}, \quad (2.24)$$

where it is used the expansion in modes around the circle: $\mathcal{L}(\phi) = -\frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \mathcal{L}_n e^{-in\phi}$.

All we have said so far concerns just gravity, the first positive attempt to introduce higher spin theories using CS formulation in 3D was done in [20], the Lie algebra used was infinite dimensional, having then and infinite tower of increasing spins fields. But there is still a further simplification in 3D, and one can construct a consistent theory with a finite number of higher spin fields, indeed we follow the results of [21], were was shown that at the linearized level the CS action reproduce the coupling to gravity of higher spin fields on AdS_3 background with the expected gauge symmetry, the constraints on the commutators of the higher spin algebra with spin 2 elements are explicitly shown. Indeed the problem as posed in [21], consists in finding the proper finite dimensional algebra using the said constraints, it happens then that $\mathfrak{sl}(N, \mathbb{R}) \oplus \mathfrak{sl}(N, \mathbb{R})$ is a very nice example of an algebra fulfilling them, the theory will have one field of each HS spin field up to N . One can also write the full non linear action that one gets by inverting the map, namely rewriting the Chern-Simon action (2.6) in terms of the frames of gravity and higher spin fields.

² To get this charge we assume Λ is not field dependent, should this not be the case we write $\delta Q(\Lambda) = -\frac{k}{2\pi} \int_{S_1} \text{tr}(\lambda(\phi)\delta A_\phi(\phi))$

2.3 $\mathfrak{sl}(3 + \dots, \mathbb{R})$

We will now move from $\mathfrak{sl}(2, \mathbb{R})$ -valued connections to the $\mathfrak{sl}(3, \mathbb{R})$ (or $\mathfrak{sl}(N, \mathbb{R})$ for any natural number N).

2.3.1 $\mathcal{W}_{3+\dots}$

In [21, 22], asymptotic AdS_3 spaces were embedded in the $\mathfrak{sl}(3, \mathbb{R})$ theory, the main result is the appearance of $\mathcal{W}_3 \times \mathcal{W}_3$ algebra as the ASA, looking at (2.23), we take $A(\phi)$ but now expanded in the $\mathfrak{sl}(3, \mathbb{R})$ basis:

$$A = \sum_{i=1}^3 l_i(\phi) L_i + \sum_{i=-2}^2 w_i(\phi) W_i, \quad (2.25)$$

where L_i correspond to the $\mathfrak{sl}(2, \mathbb{R})$ (in the principal embedding) subalgebra of $\mathfrak{sl}(3, \mathbb{R})$, The requirement of asymptotically AdS is written like $(\mathcal{A} - \mathcal{A}_{AdS})|_{\rho \rightarrow \infty} = O(1)$, this is a constraint which should be further implemented over (2.23). The reduction gives the following algebra [21, 22]:

$$\begin{aligned} \{\mathcal{L}(\phi_1), \mathcal{L}(\phi_2)\} &= -(\delta(\phi_1 - \phi_2) \mathcal{L}'(\phi_1) + 2\delta'(\phi_1 - \phi_2) \mathcal{L}(\phi_1)) - \frac{k}{4\pi} \delta'''(\phi_1 - \phi_2), \\ \{\mathcal{L}(\phi_1), \mathcal{W}(\phi_2)\} &= -(2\delta(\phi_1 - \phi_2) \mathcal{W}'(\phi_1) + 3\delta'(\phi_1 - \phi_2) \mathcal{W}), \\ \{\mathcal{W}(\phi_1), \mathcal{W}(\phi_2)\} &= -\frac{\sigma}{3} \left(2\delta(\phi_1 - \phi_2) \mathcal{L}'''(\phi_1) + 9\delta'(\phi_1 - \phi_2) \mathcal{L}''(\phi_1) \right. \\ &\quad \left. + 15\delta''(\phi_1 - \phi_2) \mathcal{L}'(\phi_1) \right) + 10\delta'''(\phi_1 - \phi_2) \mathcal{L}(\phi_1) + \frac{k}{4\pi} \delta^{(5)} \\ &\quad + \frac{64\pi}{k} (\delta(\phi_1 - \phi_2) \mathcal{L}(\phi_1) \mathcal{L}'(\phi_1) + \delta'(\phi_1 - \phi_2) \mathcal{L}^2(\phi_1)), \end{aligned} \quad (2.26)$$

2.3.2 Black holes in $\mathfrak{sl}(3 + \dots, \mathbb{R})$

As we are dealing with a theory of gravity, we can ask for the generalization of black hole solutions which might contain higher spin charges. In [23] this question was first address, and a set of requirements proposed:

- The black hole should have smooth BTZ limit, so if the chemical potential associated to a higher spin charge is send to zero the charge associated to this chemical potential will also go to zero.
- There should be an horizon, where the metric and higher spin fields close off smoothly.
- It should have a good thermodynamical properties, namely that thermodynamic quantities associated to the black hole fulfill the first law of thermodynamics.

These statements are not posed in a gauge invariant form, in fact the metric can be changed by higher spin transformation radically, namely, the horizon can be removed. Indeed this is realized as shown in [23].

To get a meaningful notion of horizon, the conditions above were translated to the invariant quantities of the gauge connections, *i.e* the holonomies. All these requirements condensate in asking the holonomies around the euclidean time circle to behave exactly as for the BTZ in $\mathfrak{sl}(2, \mathbb{R})$ context, namely, a trivial holonomy for the vielbein along the temporal direction. This is what gives in the case of the BTZ the relation between \mathcal{L} and the periodicity (the temperature). For the HS black holes this condition is kept [23](see also [24, 55]), the motivation being that this is what is giving a gauge invariant characterization of the horizon. Actually this condition as shown in the references, is giving the proper thermodynamic integrability conditions.

We now present a summary of black hole solutions available in the literature for the $\mathfrak{sl}(3, \mathbb{R})$ algebra [23, 24, 25, 28, 32, 55]. We present them as written in [25, 56]³:

Solution of [23]

$$\begin{aligned} \mathcal{A} &= b^{-1}Ab + b^{-1}db, \quad \bar{\mathcal{A}} = -b\bar{A}b^{-1} + bdb^{-1}, \quad b = e^{\rho L_0} \text{ and} \\ A &= (L_1 - \frac{2\pi}{k}\mathcal{L}L_{-1} - \frac{\pi}{2k}\mathcal{W}W_{-2})dx_+ \\ &\quad + \mu_3(W_2 - \frac{4\pi}{k}\mathcal{L}W_0 + \frac{4\pi^2}{k^2}\mathcal{L}^2W_{-2} + \frac{4\pi}{k}\mathcal{W}L_{-1})dx_-, \\ \bar{A} &= -(L_{-1} - \frac{2\pi}{k}\bar{\mathcal{L}}L_1 + \frac{\pi}{2k}\bar{\mathcal{W}}W_2)dx_- \\ &\quad - \bar{\mu}_3(W_{-2} - \frac{4\pi}{k}\bar{\mathcal{L}}W_0 + \frac{4\pi^2}{k^2}\bar{\mathcal{L}}^2W_2 + \frac{4\pi}{k}\bar{\mathcal{W}}L_1)dx_+, \end{aligned} \quad (2.27)$$

Solution of [24]

$$\begin{aligned} \mathcal{A} &= b^{-1}Ab + b^{-1}db, \quad \bar{\mathcal{A}} = -b\bar{A}b^{-1} + bdb^{-1}, \quad b = e^{\rho L_0} \text{ and} \\ A &= (l_p L_1 - \mathcal{L}L_{-1} + \Phi W_0)dx_+ + (l_D W_2 + \mathcal{W}W_{-2} - QW_0)dx_-, \\ \bar{A} &= -(l_p L_{-1} - \mathcal{L}L_1 - \Phi W_0)dx_- + (l_D W_{-2} + \mathcal{W}W_{-2} - QW_0)dx_+, \end{aligned} \quad (2.28)$$

Solution of [25]:

$$\begin{aligned} \mathcal{A} &= b^{-1}Ab + b^{-1}db, \quad \bar{\mathcal{A}} = -b\bar{A}b^{-1} + bdb^{-1}, \quad b = e^{\rho L_0} \text{ and} \\ A &= (L_1 - \frac{2\pi}{k}\mathcal{L}L_{-1} - \frac{\pi}{2k}\mathcal{W}W_{-2})d\phi + \left[\xi(L_1 - \frac{2\pi}{k}\mathcal{L}L_{-1} - \frac{\pi}{2k}\mathcal{W}W_{-2}) \right. \\ &\quad \left. + \eta(W_2 + \frac{4\pi}{k}\mathcal{W}L_{-1} + \frac{4\pi^2}{k^2}\mathcal{L}^2W_{-2} - \frac{4\pi}{k}\mathcal{L}W_0) \right] dt, \\ \bar{A} &= (L_{-1} - \frac{2\pi}{k}\bar{\mathcal{L}}L_1 - \frac{\pi}{2k}\bar{\mathcal{W}}W_2)d\phi - \left[\bar{\xi}(L_{-1} - \frac{2\pi}{k}\bar{\mathcal{L}}L_1 - \frac{\pi}{2k}\bar{\mathcal{W}}W_2) \right. \\ &\quad \left. + \bar{\eta}(W_{-2} + \frac{4\pi}{k}\bar{\mathcal{W}}L_1 + \frac{4\pi^2}{k^2}\bar{\mathcal{L}}^2W_2 - \frac{4\pi}{k}\bar{\mathcal{L}}W_0) \right] dt, \end{aligned} \quad (2.29)$$

³ There are also $\mathfrak{sl}(4, \mathbb{R})$ in [27]

Notice that the first and second solution, do not respect the BCs for which the \mathcal{W}_3 algebra in [21, 22] was obtained. So one expects asymptotic symmetry algebra of conserved charges will be different. Still in [34] a $\mathcal{W}_3 \times \mathcal{W}_3$ was associated perturbatively to the solution in [23]. In the next chapter we will focus on this statement.

2.4 $\mathfrak{hs} \times \mathfrak{hs}(\lambda)$ Theory and Coupling to Matter

As we have said before the original formulation of Vasiliev was presented with an alternative formalism, since we are using a description in terms of CS fields, we have to translate a least some results to the CS context. The original description uses a deformed oscillator algebra with auxiliary twistor variables which will be the generators of the higher spin algebra, the relevant product of this algebra is the Moyal product [12]. And the relevant fields of the Vasiliev system are three master fields: an space time 1-form W and spacetime scalars B and S_α (with a free twistor index or a form in the twistor space).

For what we are going to say it is enough to look at [57]. Here we want just to translate what things in the original formulation mean in our context:

- The Moyal product that defines the $\mathfrak{hs}(\lambda)$ algebra will be written in terms of the so called lone star product introduced first in [58], we present the rules of this product in the appendix (A).
- W contains the relevant pair of CS connections $(\mathcal{A}, \bar{\mathcal{A}})$ with values in $\mathfrak{hs}(\lambda)$, B contains a pair of master fields C and \bar{C} with values in $\mathfrak{hs}(\lambda) \cup V_0^1$, V_0^1 is the identity, and precisely the identity components are the physical degrees of freedom.
- The master field S_α does not appear at the linearized level, which is at the level we will work in here.
- The linearized Vasiliev equations are given by:

$$\begin{aligned}
 d\mathcal{A} + \mathcal{A} \wedge \star \mathcal{A} &= 0, \\
 d\bar{\mathcal{A}} + \bar{\mathcal{A}} \wedge \star \bar{\mathcal{A}} &= 0, \\
 dC + \mathcal{A} \star C - C \star \bar{\mathcal{A}} &= 0, \\
 d\bar{C} + \bar{\mathcal{A}} \star \bar{C} - \bar{C} \star \mathcal{A} &= 0.
 \end{aligned} \tag{2.30}$$

The first two equations we recognized as the flatness conditions for the CS connections, while the two last equations are the coupling to the matter fields.

3. ASYMPTOTIC SYMMETRY ALGEBRA FOR $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$ SOLUTIONS

This chapter is entirely based on [35]. We will investigate the asymptotic symmetry algebra of the $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$ black hole solutions, we will focus in the solutions presented in [23], we want to make connections with the perturbative result in [34], where $\mathcal{W}_3 \times \mathcal{W}_3$ was found. We will do it by using the canonical Dirac reduction on the affine algebra obtained by the Regge-Teitelboim method depicted in the previous chapter. We will be using a different representation for the $\mathfrak{sl}(3, \mathbb{R})$ (or any N) algebra, we will see it as the truncation of $\mathfrak{hs}[\lambda]$ to $\lambda = 3$ ($\lambda = N$). This will allow us later to go to the infinite dimensional algebra case.

The $\mathfrak{hs}(\lambda)$ algebra is described in appendix (A.2), and can be thought as constructed of the elements of $\mathfrak{sl}(2, \mathbb{R})$, we write the elements as :

$$V_{m_s}^s \quad \text{where} \quad -s+1 \leq m_s \leq s-1 \quad \text{and} \quad s = 2, 3, 4, \dots \quad (3.1)$$

And the basic product with which we construct commutators was built in [58]. Here, we want just to remind the following behavior of the commutators

$$\begin{aligned} [V_m^{s1}, V_n^{s2}] &\propto (\lambda - N) V_*^{N \leq} + V_*^{N >} \quad s1, s2 > N, \\ [V_m^{s1}, V_n^{s2}] &\propto (\lambda - N) V_*^{N \leq} \quad s1 > N \text{ or } s2 > N, \end{aligned} \quad (3.2)$$

So when $\lambda = N$ the elements V_m^s with $s \geq N$ form an ideal, so moding out by this ideal we get $\mathfrak{sl}(N, \mathbb{R})$, and the fact that one can identify the $\mathfrak{sl}(2, \mathbb{R})$ elements L_1, L_0, L_{-1} with V_1^2, V_0^2, V_{-1}^2 respectively.

Before starting we should keep in mind:

- The super index (0) in a given quantity X stands for its restriction to the Cauchy surface $X^{(0)}$. Or equivalently to its initial condition under a given flow equation.
- The symbol δ stands for an arbitrary functional variation whereas δ_Λ stands for a variation due to a residual gauge transformation Λ .
- we will forget about the coupling constant κ in front of the CS action, it can be recover in the final result by dimensional analysis.

We will focus only on the unbarred sector of the pair of connections $(\mathcal{A}, \bar{\mathcal{A}})$ taking values in $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$, and we will denote the space-time coordinates by (ρ, x_1, x_2) (So we will be free afterward to set them at convenience for example (t, ϕ) or (x_+, x_-)). As before part of the $\mathfrak{hs}(\lambda)$ gauge freedom is fixed by the choice

$$\mathcal{A}_\rho = V_0^2, \quad (\bar{\mathcal{A}}_\rho = -V_0^2). \quad (3.3)$$

The $(1, \rho)$ and $(2, \rho)$ components of the equations of motion $d\mathcal{A} + \mathcal{A}^2 = 0$ impose the form

$$\mathcal{A}_a = bA_ab^{-1}, \quad b = e^{-\rho V_0^2} \left(\bar{\mathcal{A}}_a = \bar{b}A_a\bar{b}^{-1}, \quad \bar{b} = e^{\rho V_0^2} \right), \quad (3.4)$$

with $a = 1, 2$. The remaining $(1, 2)$ components read

$$dA + A^2 = 0, \quad d \equiv dx^a \partial_a. \quad (3.5)$$

Up to this point we have twice as many variables than equations. Equation (3.5) can be thought of as:

- x_2 evolution equation for A_1 . ($\partial_2 A_1 + \dots = 0$).

where the \dots define quantities that do not involve derivatives with respect to x_2 .

From this point of view A_2 is an arbitrary source and the Cauchy surface initial condition is $A_1|_{x_2=fixed}$. The arbitrariness of the source A_2 represents an extra gauge freedom that tunes the x_2 evolution of a Cauchy data surface $A_1|_{x_2=fixed}$. Should we make the choice $A_2 = 0$, evolution is trivial and all Cauchy surfaces have the same data $A_1(x_1)$. Data $A_1(x_1)$ and $A_1(x_1) + \delta_\Lambda A_1(x_1)$ are physically inequivalent as the gauge degeneracy has been already fixed.

However, notice that one can map $\delta_\Lambda A_1(x_1)$ to an “improper” $\mathfrak{hs}(3)$ residual gauge transformation with parameter $\Lambda(x_1)$ ¹. In this way the gauge choice $A_2 = 0$ is preserved and

$$\delta_\Lambda A_1(x_1) \equiv \partial_1 \Lambda(x_1) + [A_1, \Lambda]. \quad (3.6)$$

The $x_2 = fixed$ Poisson bracket algebra reads now

$$\{\mathcal{A}_1, \mathcal{A}_\rho\}_{PB} = -\{\mathcal{A}_\rho, \mathcal{A}_1\}_{PB} = V_0^1 \delta^{(2)}. \quad (3.7)$$

Where by V_0^1 we mean the identity operator in the $\mathfrak{hs}(\lambda)$ algebra (See appendix (A)). However we are free to take $\mathcal{A}_2 = 0$ as it is compatible (first class) with (3.7). The quantity

$$G(\Gamma) \equiv \int dx_1 tr(\Gamma \mathcal{A}_1)|_{\rho=\infty} + \int dx_1 d\rho \, tr(\Gamma \mathcal{F}_{1\rho}), \quad (3.8)$$

is defined over each $x_2 = fixed$ Cauchy surface and obeys the following properties

$$\begin{aligned} \{G(\Gamma), \mathcal{A}_{1,\rho}\}_{PB} &= D_{1,\rho} \Gamma \equiv \delta_\Gamma \mathcal{A}_{1,\rho}, \\ \delta_{\mathcal{A}_1} G(\Gamma) &= - \int dx_1 d\rho \, tr(D_\rho \Gamma \delta \mathcal{A}_1), \end{aligned} \quad (3.9)$$

¹ In terms of the calligraphic components \mathcal{A} , the gauge parameter is $b\Lambda(x_1)b^{-1}$, in such a way that it preserves the $\mathfrak{hs}(3)$ gauge choice $(\mathcal{A}_\rho, \mathcal{A}_2) = (V_0^2, 0)$ (and hence it represents a “residual” gauge transformation). The gauge transformation Λ , while preserving the gauge choice and hence being “residual”, will be called “improper” if it changes the boundary data. In general it defines motion in the physical phase space. In a manner that will be explicitly shown below, these transformations define global symmetries.

under the brackets (3.7). Namely, it generates the gauge transformations on a given Cauchy surface under (3.7), and it is properly differentiable under off-shell variations $\delta\mathcal{A}_1$.

Using the results of the previous chapter we simply get again (2.23)

$$\{A_1^a(x_1), A_1^b(y_1)\}_{PB} = f_c^{ab} A_1^c \delta(x_1 - y_1) - g^{ab} \partial_{x_1} \delta(x_1 - y_1), \quad (3.10)$$

The main idea of the chapter is to analyze the phase space of $\mathfrak{sl}(3, \mathbb{R})$ CS theories with modified boundary condition. By modified we mean others than the highest weight condition used in [21, 22]. With that goal in mind, we will compute explicitly the Dirac bracket algebra with the Dirichlet boundary conditions introduced in [23] and studied in [34]. In section (3.1) we compute the fixed time Dirac bracket algebra that comes from the imposition of 6 constraints on the $\mathfrak{sl}(3, \mathbb{R})$ Kac-Moody algebra (3.10). In section (3.1.1) we recompute the same bracket algebra by use of the method of variation of the generators that was used in section (3) to compute the Kac-Moody algebra (3.10). Let us be more precise in summarizing this last result. The bracket algebra obtained by the method of variation of generators will depend on a set of integration constants that describe all possible field redefinitions of the smearing gauge parameter. As will be checked in subsection (3.1.1), for a specific choice of these integration constants this algebra will coincide with the Dirac bracket algebra reported in section (3.1).

Additionally, we must say, that there is another choice of the aforementioned integration constants that, as shown in section (3.2), define a \mathcal{W}_3 bracket algebra (up to redefinitions of the generators). In subsection (3.3) we check that such a choice of integration constants is equivalent to performing a non residual gauge transformation to the highest weight choice [21, 22]. This is also the redefinition used by the authors in [34] to arrive to a \mathcal{W}_3 symmetry transformation. Let us be more specific before entering in details. As already said and shown in subsections (3.2) and (3.3), this choice of integration constants consists of both, a redefinition of the residual gauge transformation parameters and a redefinition of the phase space parameters (the background connection). The field dependent redefinition of the residual gauge parameters to be used in this case differs with the one used in the case mentioned in the previous paragraph. This difference suggests, and we will check so, that the Dirac bracket algebra we have referred to in the last sentence of the previous paragraph is not isomorphic to \mathcal{W}_3 [25]. Accordingly, the \mathcal{W}_3 symmetry transformation, that the authors in [34] arrive to, after performing the corresponding transformations, is not acting on the original phase space of parameters (up to coordinates redefinitions) but on a different phase space given by the highest weight gauge choice [21, 22]. This last statement will be checked in section (3.2).

In section (3.4) we consider a different reduction of the $\mathfrak{sl}(3, \mathbb{R})$ phase space. In this case we classify the $\mathfrak{sl}(3, \mathbb{R})$ generators according to a diagonally embedded gravitational $\mathfrak{sl}(2, \mathbb{R})$ and impose less amount of constraints, in total 4, onto the $\mathfrak{sl}(3, \mathbb{R})$ Kac Moody algebra (3.10). By explicit computation the fixed time Dirac bracket algebra in this new phase space, is shown to be isomorphic to $\mathcal{W}_3^{(2)}$ up to first order in perturbations of the inverse of the chemical potential ν_3 .

3.1 Dirac Algebra for \mathcal{P} Phase space

We will impose 6 second class constraints (boundary conditions) onto the phase space (3.10) of 3D Chern-Simons theory with Lie algebra $\mathfrak{sl}(3, \mathbb{R})$. The reduced phase space will be called \mathcal{P} -phase space. Specifically, we compute the Dirac bracket algebra on the reduced phase space, in a Cauchy surface at fixed t_0 . The main point of this section is to show by explicit computation that this algebra is not the \mathcal{W}_3 algebra.

We start by defining what we call \mathcal{P} -phase space. First we relax the condition $A_2 = 0$ used in section (3). Besides (3.3) and (3.4), we impose the following constraints

$$\begin{aligned} A_1 &= V_1^2 + \mathcal{L}V_{-1}^2 + \mathcal{W}V_{-2}^3, \\ A_2 &= \mu_3 (V_2^3 + \text{lower components}), \end{aligned} \quad (3.11)$$

where the highest weight elements $(\mathcal{L}, \mathcal{W}, \dots)$ are arbitrary functions of (x_1, x_2) . From now on to save some notation we denote the set of all of them $(\mathcal{L}, \mathcal{W}, \dots)$ as \mathcal{M} . The boundary conditions that define the phase space of connections of the form (3.11), that we call from now on \mathcal{P} -phase space, were introduced in [23, 34] (There x_1 and x_2 are assumed light cone coordinates).

To completely precise (3.11), flatness conditions must be imposed. The flatness conditions along the generators $V_{m_s \geq -s+1}^s$ provide algebraic equations for the “lower components” in terms of $(\mathcal{M}, \partial_2 \mathcal{M})$.

$$A_2 = \mu_3 \left(V_2^3 + 2\mathcal{L}V_0^3 - \frac{2}{3}\partial_1 \mathcal{L}V_{-1}^3 + \left(\mathcal{L}^2 + \frac{1}{6}\partial_1^2 \mathcal{L} \right) V_{-2}^3 - 2\mathcal{W}V_{-1}^2 \right). \quad (3.12)$$

The remaining ones provide the x_2 -flow equations

$$\partial_2 \mathcal{L} = -2\mu_3 \partial_1 \mathcal{W}, \quad \partial_2 \mathcal{W} = \mu_3 \left(\frac{8}{3} \mathcal{L} \partial_1 \mathcal{L} + \frac{1}{6} \partial_1^3 \mathcal{L} \right), \quad (3.13)$$

which determine the \mathcal{M} out of the initial conditions $\mathcal{M}(x_1, 0)$. Solutions can be found in terms of perturbations of the chemical potential μ_3 and will have the generic form

$$\mathcal{M} = \mathcal{M}^{(0)} + \mu_3 \left(x_2 \mathcal{M}^{(1)} + \mathcal{M}_1^{(0)} \right) + O(\mu_3^2), \quad (3.14)$$

where $\mathcal{M}^{(1)}$, are local functionals of the initial conditions $\mathcal{M}^{(0)}$, $\mathcal{M}_1^{(0)}$. Notice that the integration constants $\mathcal{M}_1^{(0)}$ are just shifts in $\mathcal{M}^{(0)}$. In general we will take $\mathcal{M}_1^{(0)}$ as the most general functional of x_1 and $\mathcal{M}^{(0)}$ consistent with dimensional analysis. The explicit dependence in x_1 will play an important role.

To make things easier we start by computing the brackets on a Cauchy surface at fixed x_2 . In this case the phase space is given by the $\mathfrak{sl}(3, \mathbb{R})$ valued function of x_1 that defines the x_1 component A_1 in (3.11).

Let a generic $\mathfrak{sl}(3, \mathbb{R})$ valued function of x_1 be

$$\begin{aligned} a(x_1) &= A_{m_s}^s V_{m_s}^s = A^a V_a, \\ V_a &= (V_1^2, V_0^2, V_{-1}^2, V_2^3, V_1^3, V_0^3, V_{-1}^3, V_{-2}^3). \end{aligned} \quad (3.15)$$

We start from the Kac-Moody algebra (3.10) and proceed to impose the following 6 second class constraints

$$C^i = \left(A_1^2 - 1, A_0^2, A_2^3, A_1^3, A_0^3, A_{-1}^3 \right), \quad (3.16)$$

onto $a(x_1)$, but first we choose the integration constants $\mathcal{M}_1^{(0)}$ to be

$$\begin{aligned} \mathcal{L}_1^{(0)} &= 2\mathcal{W}^{(0)} + 2x_1\partial_1\mathcal{W}^{(0)}, \\ \mathcal{W}_1^{(0)} &= -\mathcal{L}^{(0)2} - \frac{1}{6}\partial_1^2\mathcal{L}^{(0)} - x_1\frac{1}{6}\left(16\mathcal{L}^{(0)}\partial_1\mathcal{L}^{(0)} + \partial_1^3\mathcal{L}^{(0)}\right), \end{aligned} \quad (3.17)$$

From now on, to save space we will not write down the explicit t_0 dependence but the reader should keep in mind that the full result is recovered by making the substitutions

$$\begin{aligned} \mathcal{L}^{(0)} &\rightarrow \mathcal{L}^{(0)} + \mu_3 t_0 \mathcal{W}^{(0)} + O(\mu_3^2), \\ \mathcal{W}^{(0)} &\rightarrow \mathcal{W}^{(0)} + \mu_3 t_0 \frac{1}{12} \left(16\mathcal{L}^{(0)}\partial_1\mathcal{L}^{(0)} + \partial_1^3\mathcal{L}^{(0)} \right) + O(\mu_3^2), \end{aligned} \quad (3.18)$$

at the very end.

The constraints (3.16) define the Dirac bracket

$$\{A^a(x_1), A^b(y_1)\}_D = \{A^a(x_1), A^b(y_1)\}_{PB} - \left(\{A^a, C^i\}_{PB} M_{ij} \{C^j, A^b\}_{PB} \right) (x_1, y_1), \quad (3.19)$$

in the reduced phase space with configurations $A^a = (\mathcal{L}^{(0)}, \mathcal{W}^{(0)})$.

The object $M_{ij}(x_1, y_1)$ is the inverse operator of $\{C^i(x_1), C^j(x_2)\}_{PB}$, whose non trivial components are computed to be

$$\begin{aligned} M_{12} &= \frac{1}{2}\delta_{x_1 y_1}, \quad M_{21} = -M_{12}, \quad M_{22} = \frac{1}{2}\partial_{x_1}\delta_{x_1 y_1}, \quad M_{36} = -\frac{1}{4}\delta_{x_1 y_1}, \\ M_{45} &= \frac{1}{12}\delta_{x_1 y_1}, \quad M_{46} = -\frac{1}{12}\partial_{x_1}\delta_{x_1 y_1}, \quad M_{54} = -M_{45}, \quad M_{55} = \frac{1}{24}\partial_{x_1}\delta_{x_1 y_1}, \\ M_{56} &= -\frac{1}{4}(\mathcal{L}^{(0)}\delta_{x_1 y_1} + \frac{1}{6}\partial_{x_1}^2\delta_{x_1 y_1}), \quad M_{63} = -M_{36}, \quad M_{64} = M_{46}, \quad M_{65} = -M_{56}, \\ M_{66} &= -\frac{1}{4}(\partial_{x_1}\mathcal{L}^{(0)}\delta_{x_1 y_1} + 2\mathcal{L}^{(0)}\partial_{x_1}\delta_{x_1 y_1} + \frac{1}{6}\partial_{x_1}^3\delta_{x_1 y_1}). \end{aligned} \quad (3.20)$$

It is easy to check that $M_{ij}(x_1, y_1) = -M_{ji}(y_1, x_1)$ as it should be. After some algebra (3.19) takes the explicit form

$$\begin{aligned} \{\mathcal{L}^{(0)}(y_1), \mathcal{L}^{(0)}(x_1)\}_D &= \partial_{x_1}\mathcal{L}^{(0)}\delta_{x_1 y_1} + 2\mathcal{L}^{(0)}\partial_{x_1}\delta_{x_1 y_1} + \frac{1}{2}\partial_{x_1}^3\delta_{x_1 y_1}, \\ \{\mathcal{L}^{(0)}(y_1), \mathcal{W}^{(0)}(x_1)\}_D &= 2\partial_{x_1}\mathcal{W}^{(0)}\delta_{x_1 y_1} + 3\mathcal{W}^{(0)}\partial_{x_1}\delta_{x_1 y_1}, \\ \{\mathcal{W}^{(0)}(y_1), \mathcal{W}^{(0)}(x_1)\}_D &= -\frac{1}{6}\left(16\mathcal{L}^{(0)}\partial_{x_1}\mathcal{L}^{(0)} + \partial_{x_1}^3\mathcal{L}^{(0)}\right)\delta_{x_1 y_1} - \\ &\quad \frac{1}{12}\left(9\partial_{x_1}^2\mathcal{L}^{(0)} + 32\mathcal{L}^{(0)2}\right)\partial_{x_1}\delta_{x_1 y_1} - \frac{5}{4}\partial_{x_1}\mathcal{L}^{(0)}\partial_{x_1}^2\delta_{x_1 y_1} - \\ &\quad \frac{5}{6}\mathcal{L}^{(0)}\partial_{x_1}^3\delta_{x_1 y_1} - \frac{1}{24}\partial_{x_1}^5\delta_{x_1 y_1}, \end{aligned} \quad (3.21)$$

where all the $\mathcal{L}^{(0)}$ and $\mathcal{W}^{(0)}$ in the right hand side are evaluated on x_1 . The brackets (3.21), define a \mathcal{W}_3 algebra at fixed light cone coordinate x_2 slices² for the phase space (3.11) [34, 59]. Notice that in this case, the μ_3 dependence is implicit in the fields through the redefinitions (3.18).

Now we go a step forward to compute the Dirac bracket on a Cauchy surface at fixed time t_0 . This time the constraints will look like

$$C^i = \left(A_1^2 - 1, A_0^2, A_2^3 - \mu_3, A_1^3, A_0^3 - 2\mu_3 \mathcal{L}, A_{-1}^3 + \frac{2}{3}\mu_3 \partial_1 \mathcal{L} \right), \quad (3.22)$$

and the corresponding first order in μ_3 corrections to (3.20) are

$$\begin{aligned} M_{14}^1 &= \frac{1}{6} \delta_{x_1 y_1}, \quad M_{15}^1 = -\frac{1}{6} \partial_{x_1} \delta_{x_1 y_1}, \quad M_{16}^1 = \delta_{x_1 y_1} \mathcal{L}^{(0)} + \frac{1}{4} \partial_{x_1}^2 \delta_{x_1 y_1} \\ M_{23}^1 &= -\frac{1}{2} \delta_{x_1 y_1}, \quad M_{24}^1 = \frac{1}{3} \partial_{x_1} \delta_{x_1 y_1}, \quad M_{25}^1 = -\frac{2}{3} \delta_{x_1 y_1} \mathcal{L}^{(0)} - \frac{1}{4} \partial_{x_1}^2 \delta_{x_1 y_1}, \\ M_{26}^1 &= \frac{5}{3} \delta_{x_1 y_1} \partial_{x_1} \mathcal{L}^{(0)} + \frac{7}{3} \partial_{x_1} \delta_{x_1 y_1} \mathcal{L}^{(0)} + \frac{1}{3} \partial_{x_1}^3 \delta_{x_1 y_1}, \quad M_{32}^1 = -M_{23}^1, M_{41}^1 = -M_{14}^1, \\ M_{42}^1 &= M_{24}^1, \quad M_{51}^1 = M_{15}^1, \quad M_{52}^1 = -M_{25}^1, \quad M_{56}^1 = -\frac{1}{6} \delta_{x_1 y_1} \mathcal{W}^{(0)}, \quad M_{61}^1 = -M_{16}^1, \\ M_{62}^1 &= \frac{2}{3} \delta_{x_1 y_1} \partial_{x_1} \mathcal{L}^{(0)} + \frac{7}{3} \partial_{x_1} \delta_{x_1 y_1} \mathcal{L}^{(0)} + \frac{1}{3} \partial_{x_1}^3 \delta_{x_1 y_1}, \quad M_{65}^1 = -M_{56}^1, \\ M_{66}^1 &= -\frac{1}{3} \delta_{x_1 y_1} \partial_{x_1} \mathcal{W}^{(0)} - \frac{2}{3} \partial_{x_1} \delta_{x_1 y_1} \mathcal{W}^{(0)}. \end{aligned} \quad (3.23)$$

Again it is easy to check that $M_{ij}^1(x_1, y_1) = -M_{ji}^1(y_1, x_1)$. From (3.19), (3.20) and (3.23) we compute the corresponding Dirac bracket. They can be checked to obey the compatibility property $\{C^i, \dots\}_D = 0$.

The corrections to (3.21) are given by

$$\begin{aligned} \{\mathcal{L}^{(0)}(y_1), \mathcal{L}^{(0)}(x_1)\}_D &= \dots + 2\mu_3 \partial_{x_1} \mathcal{W}^{(0)} \delta_{x_1 y_1} + 4\mu_3 \mathcal{W}^{(0)} \partial_{x_1} \delta_{x_1 y_1}, \\ \{\mathcal{L}^{(0)}(y_1), \mathcal{W}^{(0)}(x_1)\}_D &= \dots - \mu_3 \left(\frac{8}{3} \mathcal{L}^{(0)} \partial_{x_1} \mathcal{L}^{(0)} \delta_{x_1 y_1} + \frac{1}{6} \partial_{x_1}^3 \mathcal{L}^{(0)} \delta_{x_1 y_1} + \right. \\ &\quad \left. \frac{13}{3} \mathcal{L}^2 \partial_{x_1} \delta_{x_1 y_1} + \frac{4}{3} \partial_{x_1}^2 \mathcal{L}^{(0)} \partial_{x_1} \delta_{x_1 y_1} + \right. \\ &\quad \left. \frac{25}{6} \partial_{x_1} \mathcal{L}^{(0)} \partial_{x_1}^2 \delta_{x_1 y_1} + \frac{11}{3} \mathcal{L}^{(0)} \partial_{x_1}^3 \delta_{x_1 y_1} + \frac{1}{3} \partial_{x_1}^5 \delta_{x_1 y_1} \right), \\ \{\mathcal{W}^{(0)}(y_1), \mathcal{W}^{(0)}(x_1)\}_D &= \dots - \mu_3 \left(\frac{22}{3} \partial_{x_1} (\mathcal{W}^{(0)} \mathcal{L}^{(0)}) \delta_{x_1 y_1} + \partial_{x_1}^3 \mathcal{W}^{(0)} \delta_{x_1 y_1} + \right. \\ &\quad \left. \frac{44}{3} \mathcal{L}^{(0)} \mathcal{W}^{(0)} \partial_{x_1} \delta_{x_1 y_1} + \frac{10}{3} \partial_{x_1}^2 \mathcal{W}^{(0)} \partial_{x_1} \delta_{x_1 y_1} + \right. \\ &\quad \left. 4 \partial_{x_1} \mathcal{W}^{(0)} \partial_{x_1}^2 \delta_{x_1 y_1} + \frac{8}{3} \mathcal{W}^{(0)} \partial_{x_1}^3 \delta_{x_1 y_1} \right), \end{aligned} \quad (3.24)$$

and can not be reabsorbed by a general analytical redefinition at first order in μ_3

$$\mathcal{L} \rightarrow \mathcal{L} + \mu_3 \mathcal{L}_{1hom}^0, \quad \mathcal{W} \rightarrow \mathcal{W} + \mu_3 \mathcal{W}_{1hom}^0, \quad (3.25)$$

where the $(\mathcal{L}_{1hom}^0, \mathcal{W}_{1hom}^0)$ are given in the first line of (B.3). So the fixed time Dirac bracket algebra (3.24) on the phase space (3.11) is not isomorphic to \mathcal{W}_3 . However as we will see (3.11) can be embedded in a larger phase space whose constrained algebra at fixed time slices will be shown to be isomorphic to $W_3^{(2)}$.

² This is, when evolution along x_2 is considered.

3.1.1 Dirac bracket algebra in \mathcal{P} alla Regge-Teitelboim

For completeness we will recompute the Dirac bracket algebra (3.24) by use of the method of smeared variation of generators used in the computation of (3.10) in section (3).

We start by determining the set of residual (and improper) gauge transformations that map the \mathcal{P} -phase space onto itself, namely, that preserve the set of boundary conditions defining the \mathcal{P} -phase space. We ask now for the set of linear gauge transformations preserving the boundary conditions (3.11)

$$\delta A_a = \partial_{x_a} \Lambda + [A_a, \Lambda], \quad (3.26)$$

$$\Lambda = \epsilon V_1^2 + \eta V_2^3 + \text{higher components}, \quad (3.27)$$

³ where the lowest components $\{\epsilon, \eta\}$ are arbitrary functions of (x_1, x_2) . We will denote the set of lowest components $\{\epsilon, \eta\}$ by Θ . The projection along the generators $V_{m_s > -s+1}^s$ of the x_1 equation in (3.26) solves algebraically for the highest components in terms of the lowest ones Θ :

$$\begin{aligned} \Lambda(\epsilon, \eta) = & \epsilon V_1^2 - \partial_1 \epsilon V_0^2 + \left(\mathcal{L} \epsilon - 2\mathcal{W} \eta + \frac{1}{2} \partial_1^2 \epsilon \right) V_{-1}^2 + \eta V_2^3 - \partial_1 \eta V_1^3 + \\ & \left(2\mathcal{L} \eta + \frac{1}{2} \partial_1^2 \eta \right) V_0^3 - \left(\frac{2}{3} \partial_1 \mathcal{L} \eta + \frac{5}{3} \mathcal{L} \partial_1 \eta + \frac{1}{6} \partial_1^3 \eta \right) V_{-1}^3 + \\ & \left(\mathcal{W} \epsilon + \mathcal{L}^2 \eta + \frac{7}{12} \partial_1 \mathcal{L} \partial_1 \eta + \frac{1}{6} \partial_1^2 \mathcal{L} \eta + \frac{2}{3} \mathcal{L} \partial_1^2 \eta + \frac{1}{4} \partial_1^4 \eta \right) V_{-2}^3. \end{aligned} \quad (3.28)$$

Notice that the A_2 component (3.12) can be viewed as a residual gauge parameter $\Lambda(0, \mu_3)$ (as the results of [60] suggests).

The remaining x_1 equations provide variations of the gauge field parameters $\mathcal{M}(x_1, x_2)$

$$\begin{aligned} \delta_\Lambda \mathcal{L} &= \partial_1 \mathcal{L} \epsilon + 2\mathcal{L} \partial_1 \epsilon - 2\partial_1 \mathcal{W} \eta - 3\mathcal{W} \partial_1 \eta + \frac{1}{2} \partial_1^3 \epsilon, \\ \delta_\Lambda \mathcal{W} &= \partial_1 \mathcal{W} \epsilon + 3\mathcal{W} \partial_1 \epsilon + \frac{1}{6} (16\mathcal{L} \partial_1 \mathcal{L} + \partial_1^3 \mathcal{L}) \eta + \\ & \quad \frac{1}{12} (9\partial_1^2 \mathcal{L} + 32\mathcal{L}^2) \partial_1 \eta + \frac{5}{4} \partial_1 \mathcal{L} \partial_1^2 \eta + \frac{5}{6} \mathcal{L} \partial_1^3 \eta + \frac{1}{24} \partial_1^5 \eta, \end{aligned} \quad (3.29)$$

From flatness conditions and the Dirichlet boundary condition to impose, it is clear that any other component variation of the gauge fields can be deduced out of these ones. Demanding the lowest weight components (V_1^2, V_2^3) of the final A_2 connection to be fixed, determines the x_2 -flow equations

$$\partial_2 \epsilon = -\mu_3 \left(\frac{8}{3} \mathcal{L} \partial_1 \eta + \frac{1}{6} \partial_1^3 \eta \right), \quad \partial_2 \eta = 2\mu_3 \partial_1 \epsilon, \quad (3.30)$$

³ Notice that in (3.26) we have used δ and not δ_Λ . In fact we use $\delta_\Lambda A$ to denote the solution of the condition (3.26), meanwhile δ stands for an arbitrary functional variation.

which allow to solve for the gauge parameter $\Theta(x_1, x_2)$ in terms of the initial conditions $\Theta(x_1, 0)$. Again, solutions can be found in perturbations of the chemical potential μ_3

$$\Theta = \Theta^{(0)} + \mu_3 \left(x_2 \Theta^{(1)} + \Theta_1^{(0)} \right) + O(\mu_3^2), \quad (3.31)$$

where the $\Theta^{(1)}$, are local functionals of the initial conditions $\Theta^{(0)}$. The $\Theta_1^{(0)}$ are shifts of $\Theta^{(0)}$ and we will define them as general functionals of x_1 , $\mathcal{M}^{(0)}$ and $\Theta^{(0)}$ consistent with dimensional analysis, and linear in the $\Theta^{(0)}$.

Let us define our coordinates $x_1 = \frac{1}{2}(t_0 + \phi)$, $x_2 = \frac{1}{2}(-t_0 + \phi)$ and consider time evolution. This choice of coordinates identify (3.11) with the first two lines in equation (3.1) of [34] under our conventions ⁴.

The Cauchy data at a fixed time slice and the corresponding residual gauge transformations are

$$Ad\tilde{\phi} = 2A_\phi d\tilde{\phi} = A_1 dx_1 + A_2 dx_2, \quad \delta_\Lambda A = 2\delta_\Lambda A_\phi = \delta_\Lambda A_1 + \delta_\Lambda A_2, \quad (3.32)$$

where we define the angular variable $\tilde{\phi} = \frac{1}{2}\phi$.

By convenience we should choose the redefinition of generators (3.17) that was used during the explicit computation in section (3.1), namely

$$\begin{aligned} \mathcal{L}_1^{(0)} &= 2\mathcal{W}^{(0)} + 2x_1 \partial_1 \mathcal{W}^{(0)}, \\ \mathcal{W}_1^{(0)} &= -\mathcal{L}^{(0)2} - \frac{1}{6} \partial_1^2 \mathcal{L}^{(0)} - x_1 \frac{1}{6} \left(16\mathcal{L}^{(0)} \partial_1 \mathcal{L}^{(0)} + \partial_1^3 \mathcal{L}^{(0)} \right). \end{aligned}$$

By the following redefinition of residual gauge parameters

$$\begin{aligned} \epsilon_1^{(0)} &= x_1 \left(\frac{8}{3} \mathcal{L}^{(0)} \partial_1 \eta^{(0)} + \frac{1}{6} \partial_1^3 \eta^{(0)} \right), \\ \eta_1^{(0)} &= -2x_1 \partial_1 \epsilon^{(0)}, \end{aligned} \quad (3.33)$$

we get rid of all terms in the residual gauge transformation $\delta_\Lambda A$ that break periodicity under $\phi \rightarrow \phi + 2\pi$.

With the choices above, the V_{-1}^2 and V_{-2}^3 components of A become $\mathcal{L}^{(0)} + \frac{1}{2}\mu_3 t_0 \mathcal{L}^{(1)} + O(\mu_3^2)$ and $\mathcal{W}^{(0)} + \frac{1}{2}\mu_3 t_0 \mathcal{W}^{(1)} + O(\mu_3^2)$ respectively. The $(\mathcal{L}^{(1)}, \mathcal{W}^{(1)})$ are determined by the equations of motion (3.13) to be

$$\begin{aligned} \mathcal{L}^{(1)} &= 2\partial_1 \mathcal{W}^{(0)}, \\ \mathcal{W}^{(1)} &= -\frac{1}{6} \left(16\mathcal{L}^{(0)} \partial_1 \mathcal{L}^{(0)} + \partial_1^3 \mathcal{L}^{(0)} \right). \end{aligned} \quad (3.34)$$

Notice that explicit dependence in the Cauchy surface position t_0 remains in both A and $\delta_\Lambda A$. The contribution of this explicit dependence in t_0 to the charge Q is a total derivative whose integration vanishes upon imposing our periodic boundary conditions. The integrated charge, out of (2.22), for any t_0

$$Q(t_0) = \int_0^\pi d\tilde{\phi} \left(\epsilon^{(0)} \mathcal{L}^{(0)} - \eta^{(0)} \left(\mathcal{W}^{(0)} + \mu_3 \left(\frac{1}{3} \partial_1^2 \mathcal{L}^{(0)} + \frac{1}{3} \mathcal{L}^{(0)2} \right) \right) \right) + O(\mu_3^2), \quad (3.35)$$

⁴ Should we have chosen $x_1 = \phi$ and $x_2 = t$ the fixed time Dirac bracket algebra of (3.11) is seen to be \mathcal{W}_3 [25, 60].

and the variations

$$\begin{aligned} \delta_\Lambda \mathcal{L}^{(0)} = & \dots + \mu_3 \left(2\partial_1 \mathcal{W}^{(0)} \epsilon^{(0)} + 4\mathcal{W}^{(0)} \partial_1 \epsilon^{(0)} + 4\mathcal{L}^{(0)} \partial_1 \mathcal{L}^{(0)} \eta^{(0)} \right. \\ & + 3\mathcal{L}^{(0)2} \partial_1 \eta^{(0)} + 3\partial_1 \eta^{(0)} \partial_1^2 \mathcal{L}^{(0)} + \frac{11}{2} \partial_1 \mathcal{L}^{(0)} \partial_1^2 \eta^{(0)} \\ & \left. + \frac{1}{3} \partial_1^3 \mathcal{L}^{(0)} \eta^{(0)} + \frac{8}{3} \mathcal{L}^{(0)} \partial_1^3 \eta^{(0)} + \frac{1}{6} \partial_1^5 \eta \right) + O(\mu_3^2), \end{aligned} \quad (3.36)$$

$$\begin{aligned} \delta_\Lambda \mathcal{W}^{(0)} = & \dots + \mu_3 \left(-\frac{8}{3} \mathcal{L}^{(0)} \partial_1 \mathcal{L}^{(0)} \epsilon^{(0)} - \frac{13}{3} \mathcal{L}^{(0)2} \partial_1 \epsilon^{(0)} - \frac{4}{3} \partial_1^2 \mathcal{L}^{(0)} \partial_1 \epsilon^{(0)} \right. \\ & - \frac{25}{6} \partial_1 \mathcal{L}^{(0)} \partial_1^2 \epsilon^{(0)} - \frac{1}{6} \partial_1^3 \mathcal{L}^{(0)} \epsilon^{(0)} - \frac{11}{3} \mathcal{L}^{(0)} \partial_1^3 \epsilon^{(0)} - \frac{1}{3} \partial_1^5 \epsilon^{(0)} \\ & + \frac{16}{3} \mathcal{W}^{(0)} \partial_1 \mathcal{L}^{(0)} \eta^{(0)} + \frac{20}{3} \mathcal{L}^{(0)} \partial_1 \mathcal{W}^{(0)} \eta^{(0)} + \frac{38}{3} \mathcal{L}^{(0)} \mathcal{W}^{(0)} \partial_1 \eta^{(0)} \\ & \left. + \frac{10}{3} \partial_1^2 \mathcal{W}^{(0)} \partial_1 \eta^{(0)} + \frac{11}{3} \partial_1 \mathcal{W}^{(0)} \partial_1^2 \eta^{(0)} + \frac{5}{3} \mathcal{W}^{(0)} \partial_1^3 \eta^{(0)} + \partial_1^3 \mathcal{W}^{(0)} \eta^{(0)} \right) \\ & + O(\mu_3^2), \end{aligned}$$

$$\delta_\Lambda \mathcal{L}^{(1)} = \left(\delta \mathcal{L}^{(1)} \right) |_{\delta \rightarrow \delta_\Lambda},$$

$$\delta_\Lambda \mathcal{W}^{(1)} = \left(\delta \mathcal{W}^{(1)} \right) |_{\delta \rightarrow \delta_\Lambda}, \quad (3.37)$$

determine, after long but straightforward computation, the fixed time t_0 Dirac bracket algebra (3.24) by means of (2.16)⁵.

The ... in (3.36) stand for the zeroth order in μ_3 contribution, which is given by the rhs of (3.29) after substituting $(\mathcal{L}, \mathcal{W}, \epsilon, \eta)$ by $(\mathcal{L}^{(0)}, \mathcal{W}^{(0)}, \epsilon^{(0)}, \eta^{(0)})$ respectively. Remember that δ stands for arbitrary functional differential and so by $(\delta \dots) |_{\delta \rightarrow \delta_\Lambda}$ we mean to take the functional differential of ... in terms of $(\delta \mathcal{L}^{(0)}, \delta \mathcal{W}^{(0)})$ and after substitute δ by δ_Λ .

As we already said at the end of section (3.1), and stress again, the μ_3 deformation of (3.24) can not be absorbed by a field redefinition. In other words the fixed time Dirac bracket algebra (3.24) is not isomorphic to \mathcal{W}_3 .

Notice that, and we must insist on this point, a different choice of field dependent redefinition of gauge parameter than (3.33) would define a different (up to redefinition of generators) bracket algebra than the Dirac one (3.24). This is, the new bracket algebra will not correspond to the \mathcal{P} -phase space, but to a different phase space. This is what the authors in [34] have done. We will review their computations and will provide a interpretation of their results.

3.2 $\mathcal{W}_3!$

In this subsection we illustrate the issue mentioned in the previous paragraph. We will explicitly see that by using a field dependent redefinition of the gauge parameter different than (3.33) one alters the fixed time Dirac bracket algebra (of the original \mathcal{P} -phase space) to an algebra isomorphic to \mathcal{W}_3 .

⁵ ... with the substitution $(x_1, \partial_1) \rightarrow (\frac{t_0}{2} + \tilde{\phi}, \partial_{\tilde{\phi}})$ always implicitly intended.

The new choice

$$\begin{aligned}\mathcal{L}_1^{(0)} &= \dots + \mathcal{W}^{(0)}, \quad \mathcal{W}_1^{(0)} = \dots - \frac{5}{3}\mathcal{L}^{(0)2} - \frac{7}{12}\partial_1^2 \mathcal{L}^{(0)}, \\ \epsilon_1^{(0)} &= \dots - \left(\frac{8}{3}\eta^{(0)}\mathcal{L}^{(0)} + \frac{1}{4}\partial_1^2 \eta^{(0)} \right), \quad \eta_1^{(0)} = \dots + \epsilon^{(0)},\end{aligned}\tag{3.38}$$

instead of the previous ones (3.17) and (3.33), with the \dots denoting the RHS of the respective (3.17) and (3.33) expressions, defines the integrated charge

$$Q(t_0) = \int_0^\pi d\tilde{\phi} \left(\epsilon^{(0)}\mathcal{L}^{(0)} - \eta^{(0)}\mathcal{W}^{(0)} \right) + O(\mu_3^2),\tag{3.39}$$

with variations $(\delta_\Lambda \mathcal{L}^{(0)}, \delta_\Lambda \mathcal{W}^{(0)})$ given precisely as in (3.29) with $(\mathcal{L}, \mathcal{W}, \epsilon, \eta)$ substituted by the initial conditions $(\mathcal{L}^{(0)}, \mathcal{W}^{(0)}, \epsilon^{(0)}, \eta^{(0)})$.

The variations $(\delta_\Lambda \mathcal{L}^{(1)}, \delta_\Lambda \mathcal{W}^{(1)})$ are given in terms of $(\delta_\Lambda \mathcal{L}^{(0)}, \delta_\Lambda \mathcal{W}^{(0)})$, as presented in the last two lines in (3.37). Then from (2.17) one derives (3.21) which is \mathcal{W}_3 . As already stated this Poisson structure is not equivalent to the Dirac structure (3.24) mentioned before. The technical reason being the presence of the field dependent redefinition of gauge parameters (3.38) that is not equivalent to a redefinition of $(\mathcal{L}^{(0)}, \mathcal{W}^{(0)})$. As we will show this procedure is somehow violating the Dirichlet boundary conditions (3.11).

But before going on let us write down the expression for the original (V_{-1}^2, V_{-2}^3) components of the projection A_1 of A and the corresponding residual gauge parameters, $(\mathcal{L}, \mathcal{W}, \epsilon, \eta)$, in terms of the $(\mathcal{L}^{(0)}, \mathcal{W}^{(0)}, \epsilon^{(0)}, \eta^{(0)})$ for the choice (3.38).

$$\begin{aligned}\mathcal{L} &= \mathcal{L}^{(0)} + 3\mu_3\mathcal{W}^{(0)} + \mu_3 t_0 \partial_1 \mathcal{W}^{(0)} + O(\mu_3^2), \\ \mathcal{W} &= \mathcal{W}^{(0)} - \mu_3 \left(\frac{8}{3}\mathcal{L}^{(0)2} + \frac{3}{4}\partial_{x_1}^2 \mathcal{L}^{(0)} \right) - \frac{1}{12}\mu_3 t_0 \left(16\mathcal{L}^{(0)}\partial_1 \mathcal{L}^{(0)} + \partial_1^3 \mathcal{L}^{(0)} \right) + O(\mu_3^2), \\ \epsilon &= \epsilon^{(0)} - \mu_3 \left(\frac{8}{3}\eta^{(0)}\mathcal{L}^{(0)} + \frac{1}{4}\partial_{x_1}^2 \eta^{(0)} \right) + \frac{1}{12}\mu_3 t_0 \left(16\mathcal{L}^{(0)}\partial_1 \eta^{(0)} + \partial_1^3 \eta^{(0)} \right) + O(\mu_3^2), \\ \eta &= \eta^{(0)} + \mu_3 \epsilon^{(0)} - \mu_3 t_0 \partial_1 \epsilon^{(0)} + O(\mu_3^2).\end{aligned}\tag{3.40}$$

The (V_{-1}^2, V_{-2}^3) components of A are recovered by dropping the terms linear in μ_3 without t_0 dependence in the first two lines in (3.40).

3.3 No \mathcal{W}_3 !: non Residual Transformation to the Highest Weight Gauge

In the previous section we found a field dependent redefinition of the residual gauge parameter which allowed us to find a \mathcal{W}_3 algebra, in this section we will show that the process that follows the choice (3.38) in defining a \mathcal{W}_3 algebra, is equivalent to the process of performing a non residual gauge transformation⁶ that maps the \mathcal{P} -phase

⁶ This argument has been already presented by other authors in [25, 60]

space (3.11) to the highest weight gauge phase space used in [21]. In other words it is equivalent to perform a gauge transformation that changes the original boundary conditions and hence the new \mathcal{W}_3 bracket algebra, corresponds to a different phase space, not to (3.11).

Firstly, let us discuss some facts that will be useful in reaching our purpose. Let A be the space of flat connections with residual gauge transformation condition $\delta A = D_A \Lambda_A$.

Let g be an arbitrary field dependent gauge group element which is not a residual transformation of A . By performing the similarity transformation by g on both sides of $(\delta A) = D_A \Lambda_A$ we get

$$\begin{aligned} g\delta A g^{-1} &= \delta A_g - D_{A_g}(g\delta g^{-1}), \\ gD_A(\Lambda_A)g^{-1} &= D_{A_g}(g\Lambda_A g^{-1}), \end{aligned} \quad (3.41)$$

where $A_g \equiv gAg^{-1} + g\partial g^{-1}$. From (3.41) we read out the transformation law for the residual gauge parameter Λ

$$\Lambda_{A_g} = g\Lambda_A g^{-1} + g\delta g^{-1}, \quad (3.42)$$

where at this point, we are free to substitute the arbitrary differential δ by δ_{Λ_A} , the initial residual gauge transformation.

Now we notice that equations (3.13) and (3.30) are integrable at any order in μ_3 as it follows from gauge invariance [31, 34]. One way to solve them is to express the solution in terms of a gauge group element $g = g(\tilde{\mathcal{L}}, \tilde{\mathcal{W}}, \mu_3 x_2)$ that takes the highest weight connection

$$\tilde{A}_1 = V_1^2 + \tilde{\mathcal{L}}V_{-1}^2 + \tilde{\mathcal{W}}V_{-2}^3, \quad \tilde{A}_2 = 0, \quad (3.43)$$

to (3.11), via the gauge transformation law $\tilde{A} \rightarrow \tilde{A}_g \equiv A$. The element g that transforms (3.43) into (3.11) is generated at the first order in μ_3 and linear order in the algebra element by:

$$\begin{aligned} \Lambda_g &= \Lambda(\tilde{\epsilon}_g, \tilde{\eta}_g) - x_2 A_2 + O(\mu_3^2) \\ &= \Lambda(\tilde{\epsilon}_g, \tilde{\eta}_g) + \Lambda(0, -\mu_3 x_2) + O(\mu_3^2), \end{aligned} \quad (3.44)$$

with Λ , as a function of $(\tilde{\epsilon}, \tilde{\eta})$, given by (3.28) with background fields $(\tilde{\mathcal{L}}, \tilde{\mathcal{W}})$ instead of $(\mathcal{L}, \mathcal{W})$. From the second line in (3.44) it follows that Λ_g generates transformations of the kind (3.29) on the $(\tilde{\mathcal{L}}, \tilde{\mathcal{W}})$ and relate them with the new parameters $(\mathcal{L}, \mathcal{W})$ by

$$\mathcal{L} = \tilde{\mathcal{L}} - 2\mu_3 x_2 \partial_1 \tilde{\mathcal{W}} + O(\mu_3^2), \quad \mathcal{W} = \tilde{\mathcal{W}} + \mu_3 x_2 \left(\frac{8}{3} \tilde{\mathcal{L}}^2 + \frac{1}{6} \partial_1^2 \tilde{\mathcal{L}} \right) + O(\mu_3^2), \quad (3.45)$$

where we have hidden the arbitrariness $\Lambda(\tilde{\epsilon}_g, \tilde{\eta}_g)$ in (3.44), inside of the $(\tilde{\mathcal{L}}, \tilde{\mathcal{W}})$. From the x_2 flow equations (3.13) and (3.45) one is able to identify the parameters $(\tilde{\mathcal{L}}, \tilde{\mathcal{W}})$ with the initial conditions

$$\tilde{\mathcal{L}} \equiv \mathcal{L}^{(0)} + \mu_3 \mathcal{L}_1^{(0)} + O(\mu_3^2), \quad \tilde{\mathcal{W}} \equiv \mathcal{W}^{(0)} + \mu_3 \mathcal{W}_1^{(0)} + O(\mu_3^2). \quad (3.46)$$

The gauge transformation induced by g is then identified with the Hamiltonian evolution along x_2 that recovers $(\mathcal{L}, \mathcal{W})$ out of the initial conditions (3.46).

Now we can apply (3.42) to this specific case

$$\begin{aligned}\Lambda &= g\tilde{\Lambda}g^{-1} + g\delta g^{-1} \\ &= \tilde{\Lambda} + x_2(\delta A_2 - [A_2, \Lambda]) + O(\mu_3^2) = \tilde{\Lambda} + x_2\partial_2\Lambda|_{x_2=0} + O(\mu_3^2) \\ &= \tilde{\Lambda} + x_2\left(-\mu_3\left(\frac{8}{3}\tilde{\mathcal{L}}\partial_1\tilde{\eta} + \frac{1}{6}\partial_1^3\tilde{\eta}\right)V_1^2 + 2\mu_3\partial_1\tilde{\epsilon}V_2^3 + \dots\right) + O(\mu_3^2).\end{aligned}\tag{3.47}$$

Where by δ we mean the analog of the variations (3.29), and again we have hidden the arbitrariness $\Lambda(\tilde{\epsilon}_g, \tilde{\eta}_g)$ inside the parameters $\tilde{\Lambda} \equiv \Lambda(\tilde{\epsilon}, \tilde{\eta})$. The last line in (3.47), together with the x_2 flow equations (3.30), allows us to identify the parameters $(\tilde{\epsilon}, \tilde{\eta})$ with the initial conditions $(\epsilon^{(0)} + \mu_3\epsilon_1^{(0)} + O(\mu_3^2), \eta^{(0)} + \mu_3\eta_1^{(0)} + O(\mu_3^2))$. For later reference

$$\tilde{\epsilon} \equiv \epsilon^{(0)} + \mu_3\epsilon_1^{(0)} + O(\mu_3^2), \quad \tilde{\eta} \equiv \eta^{(0)} + \mu_3\eta_1^{(0)} + O(\mu_3^2).\tag{3.48}$$

After imposing (3.38), the explicit form of Λ (3.28), (3.46), (3.48) on (3.45) and (3.47), one finds the same expressions (3.38) gotten from the previous procedure for $(\mathcal{L}_1^{(0)}, \mathcal{W}_1^{(0)}, \epsilon_1^{(0)}, \eta_1^{(0)})$.

We have then proven that the process that follows the choice (3.38) in defining a \mathcal{W}_3 algebra, is equivalent to the process of performing the non residual gauge transformation (3.44) that maps the \mathcal{P} -phase space (3.11) to the highest weight gauge phase space (3.43) used in [21].

Finally, let us provide a different perspective to understand the significance of the choice of μ_3 dependence, $(\mathcal{L}_1^{(0)}, \mathcal{W}_1^{(0)}, \epsilon_1^{(0)}, \eta_1^{(0)})$, in the integration constants $(\tilde{\mathcal{L}}, \tilde{\mathcal{W}}, \tilde{\epsilon}, \tilde{\eta})$. From (3.42) it follows that the differential of charge $\delta Q \equiv \int_0^\pi d\tilde{\phi} \operatorname{tr}(\tilde{\Lambda}\delta A)$ is not invariant under a generic gauge transformation. In particular, the differential of charge for (3.43) previous to the gauge transformation g encoding the x_2 evolution, is:

$$\delta Q(\tilde{\epsilon}, \tilde{\eta}) \equiv \int_0^\pi d\tilde{\phi} \operatorname{tr}(\tilde{\Lambda}\delta\tilde{A}_1) = \int_0^\pi d\tilde{\phi} \left(\tilde{\epsilon}\delta\tilde{\mathcal{L}} - \tilde{\eta}\delta\tilde{\mathcal{W}}\right),\tag{3.49}$$

and picks up an extra μ_3 dependence after a generic μ_3 dependent non residual gauge transformation is performed. The choice (3.38) is the one that cancels, up to trivial integrations of total derivatives, the extra μ_3 dependence contribution to the final differential of charge. The final result for the transformed charge, after functional integration is performed, coincides with (3.39). This result is a consequence of the fact that the transformation g to the highest weight gauge is equivalent to perform the field dependent redefinition (3.38).

Notice that in consequence, the non residual gauge transformation g takes to a phase space (3.43) different than the \mathcal{P} -phase space (3.11). As this non residual gauge transformation g is equivalent to the choice (3.38) we have thence proven that the field dependent redefinition of residual gauge parameter (3.38) does not preserve the form of the \mathcal{P} -phase space. So, the \mathcal{W}_3 algebra obtained after performing (3.38)

does not act onto the \mathcal{P} -phase space (3.11). In consequence, the existence of the change (3.38) to a \mathcal{W}_3 algebra [34], is not in contradiction at all, with the fact that the fixed time Dirac bracket algebra, aka fixed time asymptotic symmetry algebra, computed for the \mathcal{P} -phase space (3.11) is not isomorphic to \mathcal{W}_3 .

3.4 Dirac Bracket Algebra in \mathcal{D} Phase space

In this section we try to identify a $\mathcal{W}_3^{(2)}$ Dirac bracket structure of another phase space that contains black holes [25].

Firstly, we review how to embed the \mathcal{P} -phase space (3.11) into a larger phase space. We call it \mathcal{D} -phase space after the fact we use the diagonal (\mathcal{D})⁷ embedding classification of generators to describe it⁸. Finally we compute the corresponding fixed time Dirac bracket algebra and show that it is isomorphic to $\mathcal{W}_3^{(2)}$.

First we redefine our generators as

$$\begin{aligned} J_0 &= \frac{1}{2}V_0^2, \quad J_{\pm} = \pm \frac{1}{2}V_{\pm 2}^3, \quad \Phi_0 = V_0^3, \\ G_{\frac{1}{2}}^{(\pm)} &= \frac{1}{\sqrt{8}}(V_1^2 \mp 2V_1^3), \quad G_{-\frac{1}{2}}^{(\pm)} = -\frac{1}{\sqrt{8}}(V_{-1}^2 \pm 2V_{-1}^3), \end{aligned} \quad (3.50)$$

with the non trivial commutation relations being:

$$\begin{aligned} [J_i, J_j] &= (i-j)J_{i+j}, \quad [J_i, \Phi_0] = 0, \quad [J_i, G_m^{(a)}] = \left(\frac{i}{2} - m\right)G_{i+m}^{(a)}, \\ [\Phi_0, G_m^{(a)}] &= aG_m^{(a)}, \quad [G_m^{(+)}, G_n^{(-)}] = J_{m+n} - \frac{3}{2}(m-n)\Phi_0, \end{aligned} \quad (3.51)$$

with $i = -1, 0, 1$, $m = -\frac{1}{2}, \frac{1}{2}$ and $a = \pm$. The J 's denoting the $\mathfrak{sl}(2, \mathbb{R})$ generators in the diagonal embedding. After the shift $\rho \rightarrow \rho - \frac{1}{2} \log(\mu_3)$, the space of flat connections (3.11) can be embedded into

$$\begin{aligned} A_1 &= \nu_3 \left(\sqrt{2} \left(G_{\frac{1}{2}}^{(+)} + G_{\frac{1}{2}}^{(-)} \right) - \frac{1}{\sqrt{2}} (\mathcal{G}^+ + \mathcal{G}^-) J_- - \sqrt{3} \mathcal{J} \left(G_{-\frac{1}{2}}^{(+)} + G_{-\frac{1}{2}}^{(-)} \right) \right), \\ A_2 &= 2J_+ + 2\mathcal{G}^+ G_{-\frac{1}{2}}^{(+)} + 2\mathcal{G}^- G_{-\frac{1}{2}}^{(-)} + \sqrt{6} \mathcal{J} \Phi_0 + 2\mathcal{T}' J_-, \end{aligned} \quad (3.52)$$

where $\nu_3 \equiv \mu_3^{-\frac{1}{2}}$ and

$$\begin{aligned} \mathcal{G}^+ &= \frac{\sqrt{2}}{6} \mu_3^{\frac{3}{2}} (\partial_1 \mathcal{L} + 6\mathcal{W}), \quad \mathcal{G}^- = -\frac{\sqrt{2}}{6} \mu_3^{\frac{3}{2}} (\partial_1 \mathcal{L} - 6\mathcal{W}), \\ \mathcal{J} &= \sqrt{\frac{2}{3}} \mu_3 \mathcal{L}, \quad \mathcal{T}' = -\frac{1}{6} \mu_3^2 (\partial_1^2 \mathcal{L} + 6\mathcal{L}^2). \end{aligned} \quad (3.53)$$

⁷ We call \mathcal{P} phase and \mathcal{D} because for pure mnemonic reasons, the embedding has nothing to do with the phase space and we might very well use the principal embedding to describe the phase space \mathcal{D}

⁸ We use these \mathcal{P} , \mathcal{D} prefixes to stress the difference between both phase spaces.

To obtain the previous phase space (3.11) out of (3.52), one must impose restrictions on the latter. This is, relations (3.53) imply the constraints

$$\mathcal{G}^+ - \mathcal{G}^- - \frac{1}{\sqrt{3}} \frac{1}{\nu_3} \partial_1 \mathcal{J} = 0, \quad \mathcal{T}' + \frac{1}{2\sqrt{6}} \frac{1}{\nu_3^2} \left(\partial_1^2 \mathcal{J} + \nu_3^2 \sqrt{\frac{3}{2}} \mathcal{J}^2 \right) = 0, \quad (3.54)$$

which are not compatible with the equations of motion

$$\begin{aligned} \partial_1 \mathcal{G}^\pm &= \mp \frac{\nu_3}{2\sqrt{2}} \left(6\mathcal{J}^2 \pm \sqrt{6} \partial_2 \mathcal{J} + 4\mathcal{T}' \right), \quad \partial_1 \mathcal{J} = \sqrt{3} \nu_3 (\mathcal{G}^+ - \mathcal{G}^-), \\ \partial_1 \mathcal{T}' &= -\nu_3 \left(\sqrt{3} (\mathcal{G}^- - \mathcal{G}^+) \mathcal{J} + \frac{1}{2\sqrt{2}} (\partial_2 \mathcal{G}^- + \partial_2 \mathcal{G}^+) \right), \end{aligned} \quad (3.55)$$

and hence define second class constraints on the corresponding phase space of solutions. We will not impose them, in fact they are non perturbative in ν_3 . As already mentioned, we will denote the phase space (3.52) with the prefix \mathcal{D} .

The gauge parameter of residual gauge transformations for (3.52)

$$\begin{aligned} \Lambda &= 2\Lambda_{J_+} J_+ + 2\Lambda_{G_{\frac{1}{2}}^+} G_{\frac{1}{2}}^+ + 2\Lambda_{G_{\frac{1}{2}}^-} G_{\frac{1}{2}}^- + \sqrt{6} \Lambda_{\Phi_0} \Phi_0 \\ &+ \left(-\frac{1}{2} \partial_2 \Lambda_{J_+} \right) J_0 + \left(-\mathcal{G}^+ \Lambda_{G_{\frac{1}{2}}^{(-)}} - \mathcal{G}^- \Lambda_{G_{\frac{1}{2}}^{(+)}} + 2\mathcal{T}' \Lambda_{J_+} + \frac{1}{4} \partial_2^2 \Lambda_{J_+} \right) J_- \\ &+ \left(-\sqrt{6} \mathcal{J} \Lambda_{G_{\frac{1}{2}}^+} + 2\mathcal{G}^{(+)} \Lambda_{J_+} - \partial_2 \Lambda_{G_{\frac{1}{2}}^{(+)}} \right) G_{-\frac{1}{2}}^+ \\ &+ \left(-\sqrt{6} \mathcal{J} \Lambda_{G_{\frac{1}{2}}^-} + 2\mathcal{G}^{(-)} \Lambda_{J_+} + \partial_2 \Lambda_{G_{\frac{1}{2}}^{(-)}} \right) G_{-\frac{1}{2}}^-, \end{aligned} \quad (3.56)$$

defines the variations

$$\begin{aligned} \delta_{\Lambda_{J_+}} \mathcal{T}' &= \Lambda_{J_+} \partial_2 \mathcal{T}' + 2\partial_2 \Lambda_{J_+} \mathcal{T}' + \frac{1}{8} \partial_2^3 \Lambda_{J_+}, \\ \delta_{\Lambda_{\Phi_0}} \mathcal{J} &= \partial_2 \Lambda_{\Phi_0}, \quad \delta_{\Lambda_{G_{\frac{1}{2}}^{(+)}}} \mathcal{J} = -\sqrt{6} \Lambda_{G_{\frac{1}{2}}^{(+)}} \mathcal{G}^-, \quad \delta_{\Lambda_{G_{\frac{1}{2}}^{(-)}}} \mathcal{J} = \sqrt{6} \Lambda_{G_{\frac{1}{2}}^{(-)}} \mathcal{G}^+, \\ \delta_{\Lambda_{J_+}} \mathcal{G}^{(\pm)} &= \partial_2 \Lambda_{J_+} \mathcal{G} + \frac{3}{2} \Lambda_{J_+} \partial_2 \mathcal{G}^{(\pm)} \pm \sqrt{6} \Lambda_{J_+} \mathcal{J} \mathcal{G}^\pm, \\ \delta_{\Lambda_{G_{\frac{1}{2}}^+}} \mathcal{G}^- &= \left(2\mathcal{T}' + 3\mathcal{J}^2 - \sqrt{\frac{3}{2}} \partial_2 \mathcal{J} \right) \Lambda_{G_{\frac{1}{2}}^+} - \sqrt{6} \mathcal{J} \partial_2 \Lambda_{G_{\frac{1}{2}}^+} + \frac{1}{2} \partial_2 \Lambda_{G_{\frac{1}{2}}^+}, \end{aligned} \quad (3.57)$$

and the following differential of charge in the case of x_1 evolution

$$\delta Q = \int dx_2 \text{tr} (\Lambda \delta A_2) = \int dx_2 \left(\Lambda_{J_+} d\mathcal{T} - \Lambda_{\Phi_0} d\mathcal{J} - \Lambda_{G_{\frac{1}{2}}^{(-)}} d\mathcal{G}^+ - \Lambda_{G_{\frac{1}{2}}^{(+)}} d\mathcal{G}^- \right). \quad (3.58)$$

We could now repeat the method of variation of generators done for the case of the principal embedding to this case, but instead we choose to work out the explicit computation of Dirac bracket algebra.

For the sake of brevity we will work at $t_0 = 0$, but the conclusion of this computation remains unchanged at any other fixed time slice. The difference being that the charges will carry an explicit t_0 dependence as in the previous case. At $t_0 = 0$ the Cauchy data at first order in ν_3 can be written in the form

$$\begin{aligned} A &= 2A_\phi d\tilde{\phi} = (A_{x_1} dx_1 + A_{x_2} dx_2) \\ &= \left(2J_+ + \sqrt{2}\nu_3 \left(G_{-\frac{1}{2}}^{(+)} + G_{-\frac{1}{2}}^{(-)} \right) + 2\tilde{\mathcal{G}}^{+(0)} G_{-\frac{1}{2}}^{(+)} + 2\tilde{\mathcal{G}}^{-(0)} G_{-\frac{1}{2}}^{(-)} \right. \\ &\quad \left. + \sqrt{6}\mathcal{J}^{(0)}\Phi_0 + 2\tilde{\mathcal{T}}'^{(0)}J_- \right) d\tilde{\phi} + O(\nu_3^2), \end{aligned} \quad (3.59)$$

by a choice of integration constants. Where

$$\tilde{\mathcal{G}}^{\pm(0)} = \mathcal{G}^{\pm(0)} - \frac{\sqrt{3}}{2}\nu_3\mathcal{J}^{(0)}, \quad \tilde{\mathcal{T}}'^{(0)} = \mathcal{T}'^{(0)} - \frac{1}{2\sqrt{2}}\nu_3 \left(\mathcal{G}^{+(0)} + \mathcal{G}^{-(0)} \right). \quad (3.60)$$

Again, we remind that by the super index (0) we refer to the initial conditions of the system of x_1 evolution equations (3.55). Some comments on notation are in order. Let the components of A in the $W_3^{(2)}$ basis (3.50)), be denoted again by A_a with $a = 1, \dots, 8$ and the ordering corresponding to

$$\left(J_0, J_+, J_-, \Phi_0, G_{-\frac{1}{2}}^{(+)}, G_{-\frac{1}{2}}^{(-)}, G_{-\frac{1}{2}}^{(-)}, G_{-\frac{1}{2}}^{(+)} \right). \quad (3.61)$$

At this point, we impose the four second class constraints

$$C^i = \left(A_1, A_2 - 2, A_7 - \sqrt{2}\nu_3, A_8 - \sqrt{2}\nu_3 \right), \quad (3.62)$$

on the phase space (3.59) endowed with the algebra (3.10) written in the basis (3.61). Notice that we shall not impose the second class constraints coming from (3.54). As already mentioned they are non perturbative in ν_3 .

Next, is straightforward to compute the Dirac bracket (3.19). For completeness we write down the non vanishing elements of M_{ij} in this case

$$\begin{aligned} M_{11} &= \frac{1}{8}\partial_{x_2}\delta_{x_2y_2}, \quad M_{12} = -M_{21} = -\frac{1}{2\sqrt{2}}\delta_{x_2y_2}, \quad M_{34} = -M_{43} = \frac{1}{2}\delta_{x_2y_2}, \\ M_{13} &= -M_{31} = M_{41} = -M_{14} = \frac{\nu_3}{4\sqrt{2}}\delta_{x_2y_2}, \end{aligned} \quad (3.63)$$

from where we can check explicitly by using (3.19) that $\{C^i, \dots\}_D = 0$.

The algebra in the reduced phase space will depend on ν_3 explicitly, but after implementing the change

$$\mathcal{G}_{\nu_3}^{\pm(0)} = \tilde{\mathcal{G}}^{\pm(0)} - \frac{\sqrt{3}}{2}\nu_3\mathcal{J}^{(0)}, \quad \mathcal{T}'_{\nu_3} = \tilde{\mathcal{T}}' - \frac{1}{\sqrt{2}}\nu_3(\tilde{\mathcal{G}}^{+(0)} + \tilde{\mathcal{G}}^{-(0)}), \quad (3.64)$$

we obtain the undeformed $\mathcal{W}_3^{(2)}$ algebra:

$$\begin{aligned}
\{\mathcal{T}_{\nu_3}^{\prime(0)}(y_2), \mathcal{T}_{\nu_3}^{\prime(0)}(x_2)\}_D &= \mathcal{T}_{\nu_3}^{\prime(0)} \delta_{x_2 y_2} + 2 \partial_{x_2} \mathcal{T}_{\nu_3}^{\prime(0)} \delta_{x_2 y_2} + \frac{1}{8} \partial_{x_2}^2 \delta_{x_2 y_2}, \\
\{\mathcal{J}_{\nu_3}^{(0)}(y_2), \mathcal{J}_{\nu_3}^{(0)}(x_2)\}_D &= \delta_{x_2 y_2}, \\
\{\mathcal{J}_{\nu_3}^{(0)}(y_2), \mathcal{G}_{\nu_3}^{\pm(0)}(x_2)\}_D &= \pm \sqrt{6} \mathcal{G}_{\nu_3}^{\pm(0)} \delta_{x_2 y_2}, \\
\{\mathcal{T}_{\nu_3}^{\prime(0)}(y_2), \mathcal{G}_{\nu_3}^{\pm(0)}(x_2)\}_D &= \partial_{x_2} \mathcal{G}_{\nu_3}^{\pm(0)} \delta_{x_2 y_2} + \frac{3}{2} \mathcal{G}_{\nu_3}^{\pm(0)} \partial_{x_2} \delta_{x_2 y_2} \pm \sqrt{6} \mathcal{J}_{\nu_3}^{(0)} \mathcal{G}_{\nu_3}^{\pm(0)} \delta_{x_2 y_2}, \\
\{\mathcal{G}_{\nu_3}^{+(0)}(y_2), \mathcal{G}_{\nu_3}^{-(0)}(x_2)\}_D &= - \left(2 \mathcal{T}_{\nu_3}^{\prime(0)} + 3 \mathcal{J}_{\nu_3}^{(0)2} - \sqrt{\frac{3}{2}} \partial_{x_2} \mathcal{J}_{\nu_3}^{(0)} \right) \delta_{x_2 y_2} \\
&\quad + \sqrt{6} \mathcal{J}_{\nu_3}^{(0)} \partial_{x_2} \delta_{x_2 y_2} - \partial_{x_2}^2 \delta_{x_2 y_2},
\end{aligned} \tag{3.65}$$

that agrees precisely with the signature of charges in (3.58) and the transformation laws (3.57). The most canonical form can be achieved by the usual redefinition of energy momentum tensor $\mathcal{T}_{\nu_3}^{\prime(0)} \rightarrow \mathcal{T}_{\nu_3}^{\prime(0)} + \frac{1}{2} \mathcal{J}_{\nu_3}^{(0)2}$ that makes $\mathcal{G}_{\nu_3}^{\pm(0)}$ and $\mathcal{J}_{\nu_3}^{(0)}$ primaries of weight $\frac{3}{2}$ and 1 respectively. It is then proven that the fixed time asymptotic symmetry algebra of the space of solutions (3.52) is $\mathcal{W}_3^{(2)}$ at first order in the parameter ν_3 ⁹.

Notice that (3.52) does contain the $(\mu_3, \bar{\mu}_3)$ black hole solutions [23] (of course, after performing the shift $\rho \rightarrow \rho - \frac{1}{2} \log(\mu_3)$ on them), as zero modes. Thence, both families (3.11) and (3.52) can be used to define the charges of these black holes. However, the two possibilities are not equivalent as we have already shown that (3.52) is larger than (3.11) and thence the corresponding algebras are not isomorphic. The family (3.52) is the preferred one, as for (3.11) it is impossible to define a basis of primary operators¹⁰.

We make a last comment before concluding. Notice that should we have worked with the following coordinates

$$x_1 = \frac{t + \phi}{2}, \quad x_2 = \frac{\phi}{2}, \tag{3.66}$$

all previously done remains valid, up to dependence on t_0 . This dependence only affects implicitly the $\mathcal{W}_3^{(2)}$ algebra through field redefinitions. The $\mathfrak{hs}(\lambda)$ ansätze introduced in [28], belong to (3.52) under (3.66) for the truncation to $\mathfrak{sl}(3, \mathbb{R})$ via the limit $\lambda = 3$ ¹¹. Thenceforth, in this case, the corresponding charges are not of higher spin character.

⁹ However, this should be the case at any order in ν_3 . As suggested by the non perturbative analysis reported in appendix B.2 of [25]. Notice that to compute explicitly Dirac brackets we were forced to the use of perturbation theory. For an alternative non perturbative analysis, the reader can refer to [25].

¹⁰ One can define a quasi-primary field of dimension 2, as a Virasoro subalgebra can be identified in (3.24), but the remaining generator can not be redefined in order to form a primary with respect to the Virasoro one.

¹¹ However one should keep in mind the extra shift in the coordinate $\rho \rightarrow \rho - \frac{1}{2} \log(\mu_3)$.

In our study we did not attempt to meddle with the issue of asymptotic symmetry algebras coming from generalized boundary conditions in the context of $\mathfrak{hs}(\lambda)$. We hope to report on that point in the near future.

3.5 Final Remarks

We started by analyzing the Dirac bracket algebra on the phase space of $\mathfrak{sl}(3, \mathbb{R})$ CS in principal embedding (3.11) after imposing the set of 6 constraints (3.22) on the corresponding Kac-Moody algebra (3.10) with $x_1 = \frac{t+\phi}{2}$ and $x_2 = \frac{-t+\phi}{2}$. Apart from the explicit computation, we used the method of variation of generators to cross check our result. The fixed time Dirac bracket algebra is not isomorphic to \mathcal{W}_3 .

To complete our study, and try to elucidate the apparent contradiction, we have shown that the \mathcal{W}_3 algebra that one can arrive to after a given field dependent redefinition of the smearing gauge parameter, as shown in [34] and here verified, does not act onto the original \mathcal{P} -phase space (3.11), but onto a phase space defined by a highest weight choice [21, 22, 61].

Finally, we computed the fixed time Dirac bracket algebra in phase space (3.52), containing black holes, and as expected it turned out to be isomorphic to $W_3^{(2)}$ [25, 34].

It would be necessary to address similar questions for a generic value of the deformation parameter λ . For that, analysis in perturbations of the generalized boundary conditions in the corresponding phase spaces, like the expansion in $(\mu, \bar{\mu})$ in the \mathcal{P} -phase space, or $(\nu, \bar{\nu})$ in the \mathcal{D} -phase space of the $\lambda = 3$ truncation here reviewed, could result helpful. Nevertheless we believe that an alternative and more general path to follow can be developed.

4. BLACK HOLES IN $\mathfrak{hs}(\lambda) \oplus \mathfrak{hs}(\lambda)$ THEORY

This chapter is entirely based on [28], here we construct a class of $\mathfrak{hs}(\lambda) \times \mathfrak{hs}(\lambda)$ flat connections, which fulfill the requirements to be interpreted as black holes. As a first argument, we resort to the usual relation between connections and metric like tensor fields discussed in the finite dimensional case in [21]. Next we go deeper by solving for matter coupled to the solutions found. The order of the differential equation of motion will depend on the selected representative. We show explicitly for a couple of cases and later on, prove for the complete family, that the system of differential equations is integrable in terms of the solutions for the BTZ. We will show how to compute explicitly quasi normal modes and bulk to boundary 2-point functions in the general case, and explicitly in some examples. Finally we transform our result for a particular representative to other relevant flat connections in the literature.

4.1 Black Hole Solutions

We start by writing down the generic form for the flat connections of interest:

$$\begin{aligned}\mathcal{A}_\rho &= V_0^2, & \bar{\mathcal{A}}_\rho &= -V_0^2, \\ \mathcal{A}_{t,\phi} &= bA_{t,\phi}b^{-1}, & \bar{\mathcal{A}}_{t,\phi} &= \bar{b}\bar{A}_{t,\phi}\bar{b}^{-1},\end{aligned}\tag{4.1}$$

with $b = e^{-\rho V_0^2}$ $\bar{b} = e^{\rho V_0^2}$. The generators and the relations for $\mathfrak{hs}(\lambda)$ are listed in appendix A. Let us denote our space time coordinates as (ρ, t, ϕ) and restrict our analysis to connections that obey the gauge choice (4.1) with A independent of $x_a = (t, \phi)$.

The relation between the connection and the space time tensor fields is:

$$g^{(n)} = -\frac{1}{2}tr(e^n), \quad e = \mathcal{A} - \bar{\mathcal{A}},\tag{4.2}$$

with e being the dreibein. As a starting point we remind the condition:

$$e_t|_{\rho=0} = 0,\tag{4.3}$$

required in order to have a smooth horizon at $\rho = 0$ in the spacetime tensor field $g^{(n)}$. Under (4.3) each t component in $g^{(n)}$ will have a zero at $\rho = 0$ with the appropriate order. By appropriate orders we mean those that make the corresponding reparametrization invariant quantities smooth at $\rho = 0$. For instance, $g_t^{(n)} \sim \rho^n$. In virtue of (4.1) we can rewrite (4.3) as:

$$\bar{A}_t = A_t.\tag{4.4}$$

From the flatness condition the ϕ components are constrained to be of the form:

$$A_\phi = P(A_t), \bar{A}_\phi = \bar{P}(A_t)^1, \quad (4.5)$$

where we take P and \bar{P} to be polynomials in A_t and \bar{A}_t respectively. The condition:

$$g^{(n)}(\rho) = g^{(n)}(-\rho), \quad (4.6)$$

guarantees that all the components of $g^{(n)}$ will be C^∞ in the cartesian coordinates in the plane (ρ, t) with ρ thought as the radial coordinate. Condition (4.6) ensures smoothness for the $g^{(n)}$ at $\rho = 0$. As far as euclidean conical singularity is concerned, it will be automatically excluded by requiring of the the BTZ holonomy condition [23]. See the paragraph before (4.85) for more details.

Let us identify a sufficient condition on the connections (A, \bar{A}) for (4.6) to hold. Consider the generic connections:

$$A_a = \sum_{(s, m_s)} A_{m_s}^s V_{m_s}^s, \bar{A}_a = \sum_{(s, m_s)} \bar{A}_{m_s}^s V_{m_s}^s. \quad (4.7)$$

Notice that the change ρ to $-\rho$ is equivalent to the change $V_{m_s}^s \rightarrow V_{-m_s}^s$ ².

By inserting (4.7) in (4.2), and using the properties of the \star -product, we can notice that $tr(e_a^n)$ is invariant under the combined action of $\rho \rightarrow -\rho$ and any of the following pair of \mathbb{Z}_2 transformations:

$$\text{I} : A_{m_s}^s (\bar{A}_{m_s}^s) \rightarrow A_{-m_s}^s (\bar{A}_{-m_s}^s) \quad \text{AND/OR} \quad \text{I} \times \text{II}, \quad (4.8)$$

with the \mathbb{Z}_2 II given by

$$\text{II} : \bar{A}_{-m_s}^s \rightarrow -A_{m_s}^s. \quad (4.9)$$

Transformation I together with $V_{m_s}^s \rightarrow V_{-m_s}^s$ leaves the vielbein $e_a = \mathcal{A}_a - \bar{\mathcal{A}}_a$ invariant and therefore the trace of powers of e_a . The transformation II leaves $tr(e_a^n)$ invariant but generically not the vielbein e_a .

A trivial (even) representation of $(A, \bar{A})_a$ under (4.8) is sufficient condition for (4.6). Should some components in (A, \bar{A}) not remain invariant under the \mathbb{Z}_2 I or $\text{I} \times \text{II}$, but carry a non trivial (odd) representation under any of them, then the corresponding component of the dreibein e will carry a non trivial (odd) representation too. Condition (4.6) will thus imply that traces involving an odd number of such components should vanish.

¹ It could be the case that $A_t = P(A_\phi)$ and not the other way around, but for our purposes we stick to the case written above. In fact the most general case is $A_\phi = P_\phi(A)$ and $A_t = P_t(A)$ with a generic $A \in \mathfrak{hs}(\lambda)$.

² Here we consider $s = 1, \dots, \infty$, $m_s = -2s + 1, \dots, 2s - 1$. So that under summation the indices s and m_s are mute and can be renamed without lack of rigor.

Let us analyze the particular case of the BTZ connection

$$A_t = \bar{A}_t = \frac{1}{2}a, \quad A_\phi = -\bar{A}_\phi = \frac{1}{2}a, \quad (4.10)$$

where

$$a = V_1^2 + MV_{-1}^2. \quad (4.11)$$

From now on, for simplicity, we will choose the value $M = -1$, which locates the horizon at $\rho = 0$.

The ϕ component of the pair (A, \bar{A}) remains invariant under the transformation II whereas the t component is odd. However the t component is also odd under I and so even under the composition $I \times II$. Finally, the following symmetries of the corresponding t and ϕ components of the dreibeins

$$\begin{aligned} e_t &= \frac{1}{2}(a_\rho - a_{-\rho}) \equiv a_{I \times II - \text{even}}, \\ e_\phi &= \frac{1}{2}(a_\rho + a_{-\rho}) \equiv a_{II - \text{even}}, \end{aligned} \quad (4.12)$$

imply that (4.6) holds for the connection (4.10). We can still get further information from symmetries. As e_t and e_ϕ are odd under I, any tensor field component with an odd number of t plus ϕ directions vanishes. As e_t and e_ϕ are odd and even respectively, under II, any tensor component with an odd number of t components vanish. So, finally, what said before implies that any tensor component with an odd number of ϕ directions vanish too.

What established before, holds also for generic connections. Namely:

- Any pair of connections (A, \bar{A}) that carry a trivial representation under I or $I \times II$, will define metric-like fields obeying (4.6).

We argue that (4.6) is also a necessary condition. Let us suppose that a pair (A, \bar{A}) contains a part (A^{rep}, \bar{A}^{rep}) that satisfies the conditions above, and a part $(\delta A, \delta \bar{A})$ that does not, but still defines metric like fields which are even under ρ to $-\rho$. In that case the term $(\delta \mathcal{A} - \delta \bar{\mathcal{A}})$ should be orthogonal to itself³, its powers, and powers of the generators in $(\mathcal{A}^{rep} - \bar{\mathcal{A}}^{rep})$ (This is possible to find, for example V_2^3 is orthogonal with itself and its powers). Should this not be the case the term $(\delta \mathcal{A} - \delta \bar{\mathcal{A}})$ would give contributions which are not even in ρ (based on the invariance property of the trace mentioned above). However, if $(\mathcal{A}^{rep} - \bar{\mathcal{A}}^{rep})$ contains all of the $\mathfrak{sl}(2, \mathbb{R})$ elements, $V_{0,\pm 1}^2$, it is impossible to find a set of generators in $\mathfrak{hs}(\lambda)$ that is orthogonal to every power of them. In that case, symmetry under any of the \mathbb{Z}_2 transformations in the maximal set, out of the (4.8), $(I, I \times II)$ for any (s, m_s) ⁴ is also a necessary condition for (4.6).

³ The orthogonality is meant with respect to the trace

⁴ Notice that there are many possible \mathbb{Z}_2 's. The number grows exponentially with the number of generators in $(\mathcal{A} - \bar{\mathcal{A}})$. The calligraphic letters indicate the full connection, ρ component and ρ dependence included.

At this point we specify our class of connections:

$$\begin{aligned} A_t &= \bar{A}_t = P_t(a), \\ A_\phi &= \frac{1}{2}a + P_\phi(a), \quad \bar{A}_\phi = -\frac{1}{2}a + \bar{P}_\phi(a), \end{aligned} \quad (4.13)$$

with P_t , P_ϕ and \bar{P}_ϕ being arbitrary traceless polynomials of the form

$$\begin{aligned} P_t &= \sum_{i=0}^M \nu_i (a^{2i+1} - \text{trace}), \\ P_\phi &= \sum_{i=0}^N \mu_{i+3} (a^{2i+2} - \text{trace}), \quad \bar{P}_\phi = \sum_{i=0}^M \bar{\mu}_{i+3} (a^{2i+2} - \text{trace}). \end{aligned} \quad (4.14)$$

Notice (4.14) obeys (4.3) and that P_t and P_ϕ are selected in such a way that $g_{t\phi} = 0$. The components $g_{\rho t}$ and $g_{\rho\phi}$ vanish too. In particular (4.14) reduce to the non rotating BTZ $_{M=-1}$ connection in the limit $\nu_0 = \frac{1}{2}$, $\nu_{i>0} = 0$, and vanishing μ_i , $\bar{\mu}_i$. Now:

- The transformations of a , the corresponding deformation polynomials ($P_\phi(a)$, $\bar{P}_\phi(a)$, $P_t(a)$) and the ρ components $\pm V_0^2$ under I in (4.8), are odd, even, odd and even respectively.
- In virtue of properties of the \star -product, the traces with odd numbers of a and $P_t(a)$ with any number of insertions of V_0^2 and $(P_\phi(a), \bar{P}_\phi(a))$, vanish, and so all non vanishing traces are even under I and henceforth even under $\rho \rightarrow -\rho$.

We conclude that the ansätze (4.14) give rise to spacetime tensor fields that obey (4.6). In fact we explicitly checked (4.6) to hold up to arbitrary higher order n .

In the near horizon expansion of $g^{(2)}$, the line element defined by (4.2), will look like:

$$d\rho^2 - \frac{4}{T^2} \rho^2 dt^2 + \dots = \rho^* dv^2 + \frac{1}{2} d\rho^* dv + \dots, \quad (4.15)$$

with $v = t - \frac{T}{2} \log(\rho) + \dots$ and $\rho^* = \frac{4}{T^2} \rho^2 + \dots$ being coordinate redefinitions that are going to be useful later on when analyzing fluctuations. The \dots denoting higher orders corrections in ρ . The temperature:

$$T(P_t) \equiv \frac{1}{\sqrt{\frac{1}{2} \text{tr}([P_t(a), V_0^2]^2)}},^5 \quad (4.16)$$

defines the thermal periodicity under $t \rightarrow t + \pi T i$.

⁵ From the positiveness of the traces $\text{tr}(V_{2m_s+1}^{2s} V_{2m_s-1}^{2s})$, see (A.10), in the interval $0 < \lambda < 1$ and the fact we have chosen odd powers of a in P_t it follows that the quantity inside the roots in (4.17) and (4.19) is a sum of positive defined quantities and hence positive defined. We stress that we restrict our study to the interval $0 < \lambda < 1$.

We will focus our study in the cases $\nu_0 = \frac{1}{2}$, $\nu_{i>0} = 0$. These are solutions that obey the usual BTZ holonomy-smoothness condition as the temporal component of the connection coincides with the BTZ one with $M = -1$. This implies that not only the eigenvalues of the time component of connection are the same as $\text{BTZ}_{M=-1}$, but also that the holonomy around the contractible euclidean time cycle coincides with the BTZ case, since the euclidean periodicity, determined by the temperature $T(\frac{1}{2}a) = 2$, is the same as for the $\text{BTZ}_{M=-1}$.

However before going on, let us comment on the possibility of arbitrary ν_i . The euclidean smoothness condition are:

$$e^{\pi i T(P_t)P_t(a)} \sim V_0^1. \quad (4.17)$$

To solve for (4.17) we use the fact that $\pi i P(a)$, with $P(a)$ an arbitrary polynomial of a with arbitrary integer coefficients, are known to exponentiate to V_0^1 in the region $0 < \lambda < 1$, see [62].

Then relations (4.17) reduce to find out the ν_i such that $\nu_i T(P_t)$ are integers. To study this quantization conditions it is useful to write down P_t in the basis

$$a_{\perp}^{s-1} \equiv \frac{1}{N_s} \sum_{t=0}^{s-1} (-1)^t \binom{s-1}{t} V_{s-1-2t}^s \sim (a_{\perp}^{s-1})|_{V_{m_t}^t \rightarrow 0}, \quad (4.18)$$

where N_s is a normalization factor, chosen in such a way that: $\text{tr}((a_{\perp}^{s-1})^2) = 1$. We get thus

$$P_t(a) = \sum_{s=0}^{\infty} \nu_{\perp s} \frac{a_{\perp}^{2s+1}}{\sqrt{\frac{1}{2} \text{tr}([a_{\perp}^{s-1}, V_0^2]^2)}}, \quad \nu_{\perp}^s = M^{si} \nu_i. \quad (4.19)$$

where the linear transformation matrix M is upper triangular. In the appendix B we present the explicit form for M , (C.1), for the case $\mu_{2i+1} \neq 0$, with $i = 0, \dots, 4$. An important property to use is that the eigenvalues (the diagonal elements) of M can be checked to be larger or equal than 1 in the range $0 < \lambda < 1$ until arbitrary large i .

The desired quantization conditions can be written as:

$$\nu_i T(P_t) = (M^{-1})_{is} \cos \theta^s = n_i, \quad (4.20)$$

with $\cos \theta^s \equiv \frac{\nu_{\perp}^s}{\sqrt{\sum_s (\nu_{\perp}^s)^2}}$ and n_i an arbitrary integer. The condition for the quantization relation (4.20) to admit solutions is:

$$\sum_{s=1}^{\infty} (M \cdot n)^{s^2} = 1. \quad (4.21)$$

In appendix (C.1) we show that the property of the eigenvalue of M mentioned above excludes the presence of other solutions to the consistency condition (4.21) in the region $0 < \lambda < 1$, apart from the trivial one $n_0 = 1$ ($\nu_0 = \frac{1}{2}$, $\nu_{i>0} = 0$). Here

we just continue with the cases that are continuously linked to the BTZ connection in the limit $\mu_i, \bar{\mu}_i$ to zero. Namely $\nu_0 = \frac{1}{2}$, $\nu_{i>0} = 0$. The requirement of the BTZ holonomy condition will guarantee the absence of any possible conical singularity in the tensor like fields as the dreibein itself is thermal periodic.

Generically (4.14) will define asymptotically Lifshitz metrics with critical exponent $z < 1$, except for the cases in which the contributions out of the deformation parameters $\mu_i, \bar{\mu}_i$ will not provide ρ dependence. An example being when $\bar{\mu}_i = 0$ (or $\mu_i = 0$) in which case the only contribution to the $g_{\phi\phi}$ comes at quadratic order in μ_i (or $\bar{\mu}_i$) but it is independent of ρ due to the cyclic property of the trace. In those cases the metric becomes asymptotically AdS.

To summarize, there will be metrics of two classes:

- Generically Lifshitz metric with $z < 1$.
- AdS metrics when $\mu_{2i} = 0$ (or $\bar{\mu}_{2i} = 0$).

This classification relies on the definition (4.2). For instance the line elements coming from (4.2) for the cases $\mu_3 \neq 0$, $\bar{\mu}_3 = -\mu_3 \neq 0$ and $\bar{\mu}_3 = \mu_3 \neq 0$ looks like :

$$\begin{aligned}
ds_{(\mu_3, 0)}^2 &= d\rho^2 - \sinh^2 \rho \, dt^2 + \left(\cosh^2 \rho + \frac{16(\lambda^2 - 4)}{15} \mu_3^2 \right) d\phi^2, \\
ds_{(\mu_3, -\mu_3)}^2 &= d\rho^2 - \sinh^2 \rho \, dt^2 + \frac{1}{30} (12(\lambda^2 - 4) \mu_3^2 \cosh(4\rho) \\
&\quad + 5(4(\lambda^2 - 4) \mu_3^2 + 3 \cosh(2\rho) + 3)) d\phi^2, \\
ds_{(\mu_3, \mu_3)}^2 &= d\rho^2 - \sinh^2 \rho \, dt^2 + \frac{1}{5} \cosh^2(\rho) (-8(\lambda^2 - 4) \mu_3^2 \cosh(2\rho) \\
&\quad + 8(\lambda^2 - 4) \mu_3^2 + 5) d\phi^2.
\end{aligned} \tag{4.22}$$

The first line element in (4.22) behaves asymptotically as AdS_3 and shows a smooth horizon at $\rho = 0$, while the last two cases are Lifshitz metrics with dynamical critical exponent $z = \frac{1}{2} < 1$. Should we have turned on a higher spin μ deformation, the parameter z would have decreased like $z = \frac{1}{4}, \frac{1}{8}, \dots$. The bulk of the present paper, section 3, will be devoted to the study of matter fluctuations around these backgrounds, which, of course, are not just gravitational but involve also higher spin fields turned on. This analysis will confirm the expectation that these backgrounds truly describe BH, through the “dissipative” nature of matter fluctuations we will find.

Before closing this section, we make contact (perturbatively in μ_3) with other relevant backgrounds studied in the literature recently. More precisely, we look for static gauge parameters $(\Lambda, \bar{\Lambda})$ (independent of $x_{1,2}$), that transform (4.14) to the GK [23] and BHPTT [25, 60] backgrounds. Notice that these gauge transformations will not change the eigenvalues of the components $(A_{1,2}, \bar{A}_{1,2})$ of the connections

because they are just similarity transformations. The two classes of backgrounds we want to relate ours, are described by the following connections:

$$\begin{aligned} A_1 &= V_1^2 + \mathcal{L}V_{-1}^2 + \mathcal{W}V_{-2}^3 + \mathcal{Z}V_{-3}^4 + \dots, & A_2 &= \sum_{i=0}^{\infty} \mu_{i+3} (A_1^{i+2} - \text{traces}), \\ \bar{A}_1 &= V_{-1}^2 + \bar{\mathcal{L}}V_1^2 + \bar{\mathcal{W}}V_2^3 + \bar{\mathcal{Z}}V_3^4 + \dots, & \bar{A}_2 &= \sum_{i=0}^{\infty} \bar{\mu}_{i+3} (\bar{A}_1^{i+2} - \text{traces}). \end{aligned} \quad (4.23)$$

Our parameters $(\mu_i, \bar{\mu}_i)$ will be identified precisely with the "chemical potentials" in (4.23). In our approach the charge-chemical potential relations [23, 63] are determined a priori by the condition $\nu_0 = \frac{1}{2}$, $\nu_{i>0} = 0$. Namely, after applying the gauge transformations $(\Lambda, \bar{\Lambda})$ the charges \mathcal{L} , \mathcal{W} and \mathcal{Z} will be already written in terms of the chemical potentials $(\mu_i, \bar{\mu}_i)$. In this way one can generate GK, and BHPTT ansätze with more than one $(\mu_i, \bar{\mu}_i)$ turned on, and with the holonomy conditions already satisfied. However, with the choice $\nu_0 = \frac{1}{2}$, $\nu_{i>0} = 0$ one can only reach branches that are smoothly related to the BTZ. case

Taking $x_1 = x_2 = x_+$ and $x_2 = x_{\bar{1}} = x_-$, we recover the GK background, whereas for $x_1 = x_{\bar{1}} = \phi$ and $x_2 = x_2 = t$ we get BHPTT.

For later use, we write down the particular gauge transformations that takes the representative with non vanishing $\mu_3 = -\bar{\mu}_3$ into the wormhole ansatz for GK's case. They read, respectively, to leading order in $\mu_3 = -\bar{\mu}_3$:

$$\begin{aligned} \Lambda_{GK} &= \mu_3 \left(-\frac{5}{3} e^{-\rho} V_{-1}^3 + e^{\rho} V_1^3 \right) + \text{commutant of } a_{\rho} + O(\mu_3^2), \\ \bar{\Lambda}_{GK} &= \mu_3 \left(e^{\rho} V_{-1}^3 - \frac{5}{3} e^{-\rho} V_1^3 \right) + \text{commutant of } a_{-\rho} + O(\mu_3^2). \end{aligned} \quad (4.24)$$

The holonomy conditions are satisfied a priori and so the corresponding "charge-chemical potential" relations are as follows:

$$\mathcal{L} = \bar{\mathcal{L}} = -1 + O(\mu_3^2), \quad \mathcal{W} = -\bar{\mathcal{W}} = \frac{8}{3} \mu_3 + O(\mu_3^3), \quad \mathcal{Z} = \bar{\mathcal{Z}} = O(\mu_3^2), \quad \dots \quad (4.25)$$

For BHPTT, namely when the "chemical potentials" are turned on along the t direction and the asymptotic symmetry algebra is the undeformed $\mathcal{W}_{\lambda} \times \mathcal{W}_{\lambda}$ [60, 64], they are given by:

$$\begin{aligned} \Lambda_{BHPTT} &= 2\Lambda_{GK} + O(\mu_3^2), \\ \bar{\Lambda}_{BHPTT} &= 2\bar{\Lambda}_{GK} + O(\mu_3^2). \end{aligned} \quad (4.26)$$

In this case the relations "charge-chemical potential" are:

$$\mathcal{L} = \bar{\mathcal{L}} = -1 + O(\mu_3^2), \quad \mathcal{W} = -\bar{\mathcal{W}} = \frac{16}{3} \mu_3 + O(\mu_3^3), \quad \mathcal{Z} = \bar{\mathcal{Z}} = O(\mu_3^2). \quad (4.27)$$

We will apply later these transformations to the matter fluctuations in the $\bar{\mu}_3 = -\mu_3 \neq 0$ background in (4.14).

4.2 Coupling of Matter

In this subsection we show how to obtain the differential equations for the scalar fluctuations over the backgrounds (4.14). Firstly, we review how this works for the $\text{BTZ}_{M=-1}$ case. This will allow us to identify a strategy for our cases (4.14). We will focus just on one of the scalars of the Vasiliev system, the treatment for the other scalar is completely analogous.

As mentioned in the introductory chapter, the master field C that contains the physical degree of freedom coupled to a generic background connections $(\mathcal{A}, \bar{\mathcal{A}})$ is ruled by the equation:

$$dC + \mathcal{A} \star C - C \star \bar{\mathcal{A}} = 0 \quad \text{with} \quad C = \sum_{s=1}^{\infty} C_{m_s}^s V_{m_s}^s, \quad (4.28)$$

whose formal solution and its corresponding transformation law under left multiplication $(g, \bar{g}) \rightarrow (e^\Lambda g, e^{\bar{\Lambda}} \bar{g})$, are, respectively:

$$C = g \mathcal{C} \bar{g}^{-1} \quad \text{and} \quad C_{(\Lambda, \bar{\Lambda})} = e^\Lambda C e^{-\bar{\Lambda}}, \quad (4.29)$$

where $d\mathcal{C} = 0$ and $\mathcal{C} = \sum_m C_m^s V_m^s$.

The trace part of the master field C and its transformation law are also:

$$C_0^1 = (C)|_{V_0^1} \quad \text{and} \quad C_{0(\Lambda, \bar{\Lambda})}^1 = \left(e^{(\Lambda - \bar{\Lambda})} C \right)|_{V_0^1}. \quad (4.30)$$

The integration constant \mathcal{C} is evaluated in the limit $C|_{g \rightarrow 1}$. In our cases (4.14) g goes to 1 at the point $(\rho, x_a) = 0$. However notice that this point is located at the horizon $\rho = 0$ of (4.14) and, as we shall see, many of the components of the master field C will diverge there.

Our aim is to “fold” (4.28) for our ansätze (4.14) with $\nu_0 = \frac{1}{2}$, $\nu_{i>0} = 0$. By “folding” we mean the process of expressing every $C_{m_s}^s$ in terms of C_0^1 and its derivatives, and finally to obtain a differential equation for C_0^1 . For such a purpose we start by reviewing how this process works for the simplest case, $\text{BTZ}_{M=-1}$, and in doing so we will discover how to fold the matter fluctuations in the case of our backgrounds (4.14).

We start by proving that for $\text{BTZ}_{M=-1}$ every higher spin component $C_{m_s}^s$, can be expressed in terms of ∂_\pm derivatives of C_0^1 and C_0^2 . Using the explicit forms for g and \bar{g} in this case:

$$C = e^{-a_\rho x_+} \mathcal{C}(\rho) e^{-a_- \rho x_-}. \quad (4.31)$$

It is easy to see that:

$$\partial_\pm C_0^1 = -(a_\pm \rho C)|_{V_0^1} \sim -(e^{\pm \rho} C_1^2 - e^{\mp \rho} C_{-1}^2), \quad (4.32)$$

from where (C.15) of the Appendix D is immediate, after taking the trace (A.10).

Now we can repeat the procedure at second order in \pm derivatives of C_0^1 . At this stage we can write down three combinations:

$$\partial_+^2, \partial_-^2, \partial_{+-}^2,$$

which would generate the following quadratic relations inside the trace element:

$$a_\rho^2 = \tilde{V}_0^1 + e^{2\rho}V_2^3 - 2V_0^3 + e^{-2\rho}V_{-2}^3, \quad (4.33)$$

$$a_{-\rho}^2 = \tilde{V}_0^1 + e^{-2\rho}V_2^3 - 2V_0^3 + e^{2\rho}V_{-2}^3, \quad (4.34)$$

$$a_\rho a_{-\rho} = \cosh 2\rho(\tilde{V}_0^1 - 2V_0^3) - 2\sinh 2\rho V_0^2 + V_2^3 + V_{-2}^3, \quad (4.35)$$

where $\tilde{V}_0^1 = \frac{(\lambda^2-1)}{3}V_0^1$.

Equations (4.33), (4.34) and (4.35), allow to write down C_{-2}^3 , C_0^3 and C_2^3 in terms of

$$(\partial_+^2 C_0^1, \partial_-^2 C_0^1, \partial_{+-}^2 C_0^1, C_0^2),$$

so that one arrives to the relations (C.17) and (C.19).

Proceeding this way, we see that at the level $s = 3$ we can still use first derivatives acting on C_0^2 :

$$\partial_+ C_0^2 = -(V_0^2 a_\rho C)|_{V_0^1} \quad \text{and} \quad \partial_- C_0^2 = -(a_{-\rho} V_0^2 C)|_{V_0^1}. \quad (4.36)$$

Then, if we use:

$$V_0^2 a_\rho = -\frac{1}{2}(e^\rho V_1^2 + e^{-\rho} V_{-1}^2) - e^{-\rho} V_{-1}^3 + e^\rho V_1^3, \quad (4.37)$$

$$a_{-\rho} V_0^2 = \frac{1}{2}(e^{-\rho} V_1^2 + e^\rho V_{-1}^2) - e^\rho V_{-1}^3 + e^{-\rho} V_1^3, \quad (4.38)$$

on both equations in (4.36), together with (4.32), we get the spin three components $C_{\pm 1}^3$ in terms of:

$$(\partial_+ C_0^1, \partial_- C_0^1, \partial_+ C_0^2, \partial_- C_0^2),$$

as shown in (C.18).

Now we show how this process of reduction works at any spin level s . First we remind some useful properties of the lonestar product. Let us start by the generic product

$$V_{m_1}^{s_1} \star V_{m_2}^{s_2},$$

that will reduce to a combination of the form:

$$V_{m_1+m_2}^{s_1+s_2-1} + \dots + V_{m_1+m_2}^{s_1+s_2-1-j} + \dots + V_{m_1+m_2}^{|m_1+m_2|+1}, \quad (4.39)$$

where we are not paying attention to the specific coefficients, which will be used in due time. The index j goes from 0 to $s_1 + s_2 - 2 - |m_1 + m_2|$. From (4.39) it follows

that the products: $V_{m_1}^{s_1} \star a$ and $a \star V_{m_1}^{s_1}$, with $a = V_1^2 - V_{-1}^2$, will contain combinations of the form:

$$V_{m_1+1}^{s_1+1} + V_{m_1-1}^{s_1+1} + \dots, \quad (4.40)$$

where the \dots stand for lower total spin s contributions. For our purposes only the highest total spin generators are relevant.

Furthermore, for any chain of $2s - 1$ generators with even spin $2s$ and even projections, $\sum_{m=-s+1}^{s-1} V_{2m}^{2s} + \dots$, further left or right multiplication by a will change it into a chain of $2s$ generators $\sum_{m=-s}^{s-1} V_{2m+1}^{2s+1} + \dots$ at the next spin level $2s + 1$. As a consequence, arbitrary powers of a look like:

$$a^{2s} = \sum_{m=-s}^s V_{2m}^{2s+1} + \dots \quad \text{and} \quad a^{2s+1} = \sum_{m=-s-1}^s V_{2m+1}^{2s+2} + \dots, \quad (4.41)$$

From (4.30) and (4.31), it follows that each ∂_{\pm} derivative acting on C_0^1 is equivalent to a left or right multiplication by $-a_{\pm\rho}$ inside the trace. In particular, taking $2s$ of these derivatives on C_0^1 is equivalent to take $2s$ powers of $\pm a_{\pm\rho}$ inside the trace.

The number of different derivatives of order $2s$ denoted by: ∂_{\pm}^{2s} is $2s + 1$. This number coincides precisely with the number of components with total spin $= 2s + 1$ in the first power of (4.41). So one can use the $2s + 1$ relations:

$$\partial_{\pm}^{2s} C_0^1 = (a_{\pm\rho}^{2s} C) \Big|_{V_0^1}, \quad (4.42)$$

to solve for $2s + 1$ components of C :

$$[C_{2m}^{2s+1}] \text{ with } m = -s, \dots, s, \quad (4.43)$$

in terms of components with lower total spin and their \pm derivatives.

One can always solve equations (4.42) in terms of (4.43) because the set of symmetrized powers of $a_{\pm\rho}^{2s}$ (more precisely, their components with the highest total spin) will generate a basis for the $2s + 1$ dimensional space generated by:

$$[V_{2m}^{2s+1}] \text{ with } m = -s - 1, \dots, s.$$

In order to prove this statement, we take the large ρ limit. In this limit a given symmetric product a_{\pm}^{2s} with $2m_+$ plus signs and $2m_- = 2(s - m_+)$ minus signs reduces to a single basis element $V_{2(m_+-m_-)}^{2s}$. So, the set of all possible symmetric products a_{\pm}^{2s} span an $2s + 1$ -dimensional vector space. Consequently the system of equations (4.42) is non-degenerate.

Similarly, increasing the spin by one, one can solve the $2s + 2$ relations:

$$\partial_{\pm}^{2s+1} C_0^1 = -(a_{\pm\rho}^{2s+1} C) \big|_{V_0^1}, \quad (4.44)$$

for the $2s + 2$ components

$$[C_{2m+1}^{2s+2}] \text{ with } m = -s - 1, \dots, s, \quad (4.45)$$

in terms of lower spin components and their \pm derivatives.

Summarizing, what we have done is to use the identities:

$$\partial_+ = -a_{\rho} \star_L, \quad \partial_- = -a_{-\rho} \star_R, \quad (4.46)$$

with left \star_L and right \star_R multiplication inside any trace. Notice that in Fourier space $(-i\partial_t, -i\partial_{\phi}) = (w, k)$ the master field (4.31) is an eigenstate of the operators on the right hand side of (4.46). This will turn out to be a crucial observation, and It will be useful for later purposes, but for now we just use (4.46) to solve for every component of $C_{m_s}^s$ with (s, m_s) being points in a “semi-lattice” with origin $(1, 0)$ and generated by positive integral combinations of basis vectors $(2, 1)$ and $(2, -1)$. From now on we will refer to this particular “semi-lattice” as I and to the corresponding set of components of the master field C in It as C^I .

In exactly the same manner one can show how the set of powers

$$a_{\rho}^{s+} V_0^2 a_{-\rho}^{s-}, \quad (4.47)$$

with $s = s_+ + s_- + 1$ spans the complementary “semi-lattice” of spin $s + 1$ and projection $m_s = -s + 1, -s + 3, \dots, s - 3, s - 1$ generators. Namely the “semi-lattice” with origin at $(2, 0)$ and positive integral combinations of $(2, 1)$ and $(2, -1)$. We refer to as II , and the corresponding components of the master field C , C^{II} . More in detail, this means that we can solve the s relations:

$$\partial_+^{s+} \partial_-^{s-} C_0^2 = (-1)^{s_++s_-} (a_{-\rho}^{s-} V_0^2 a_{\rho}^{s+} C) \big|_{V_0^1}, \quad (4.48)$$

for the set of components in C^{II} with highest spin $= s + 1$ and projections $m_s = -s + 1, -s + 3, \dots, s - 3, s - 1$.

- In conclusion, equations (4.42)-(4.45) and (4.48) allow to solve for every components of C^I and C^{II} in terms of C_0^1 and C_0^2 and their derivatives along \pm directions.

Finally, the V_0^1 - $d\rho$ component of (4.28) gives $C_0^2 \sim \partial_{\rho} C_0^1$ and the V_0^2 - $d\rho$ component of (4.28) will determine the differential equation $D_2 C_0^1 = 0$ with

$$D_2 = \square - (\lambda^2 - 1), \quad (4.49)$$

being the Klein Gordon operator in the $\text{BTZ}_{M=-1}$ background, for a scalar field with mass squared $\lambda^2 - 1$.

Now we go back to our case $\nu_0 = \frac{1}{2}$ $\nu_{i>0} = 0$. Here the t component of (4.28) is the same as for the $\text{BTZ}_{M=-1}$ case and so we use it as before

$$\partial_t C_{m_s+1}^{s-1} = C_{m_s}^s + C_{m_s+2}^s + \dots, \quad (4.50)$$

to solve for the highest spin, with the lowest spin projection components (s, m_s) . The dots refer to components with lower total spin and we have omitted precise factors. That is, we solve for all components in C^I and C^{II} in terms of the line of highest weight and its contiguous next-to-highest weight components, namely:

$$C_s^{s+1} \text{ and } C_s^{s+2} \text{ with } s = 0, \dots, \infty. \quad (4.51)$$

Next, $\partial_\phi \sim a^{1+\tilde{s}_{Max}} +$ lower powers, and therefore from (4.41) one can prove that the use of the $d\phi$ component of the equations (4.28) reduces the set of independent elements in (4.51) to:

$$C_s^{s+1} \text{ and } C_s^{s+2} \text{ with } 0 \leq s \leq s_{max}, \quad (4.52)$$

with $s_{max} + 1$ being at most $\tilde{s}_{max} + 1$, the maximum value of the power in the polynomials $(P(a), \bar{P}(a))$, that determines the ϕ component of the connections (A_ϕ, \bar{A}_ϕ) . Notice that for some configurations in (4.14) there are degeneracies and the number of independent components decreases in those cases. In fact s_{max} determines the degree of the differential equation for C_0^1 (or equivalently the number of ρ -components one has to use to close the system) to be given by $2(s_{max} + 1)$, after the ρ components of the equations of motion are imposed.

4.2.1 Solving the matter equations of motion

In this subsection we show how to proceed for the simplest cases, and later on we prove in general that the equations of motion for scalars in (4.14), can be expressed in terms of simpler building blocks. Let us start by explicitly exhibiting the solutions for matter fluctuations in the case of the backgrounds with $\mu_3 \neq 0$. Firstly, we determine the differential equation for C_0^1 by using the procedure outlined in the last paragraph of the previous section. In this case $s_{max} = 1$ and we get a differential equation for C_0^1 with degree $2(s_{max} + 1) = 4$ in ρ . It is convenient to Fourier transform from (ϕ, t) to (k, ω) for the fields C_m^s :

$$C_m^s[\rho, t, \phi] = e^{i\omega t} e^{ik\phi} C_m^s[\rho]. \quad (4.53)$$

The final form of the equation for C_0^1 is given in (C.13), here we will be somewhat schematic. After the change of coordinates $\rho = \tanh^{-1}(\sqrt{z})^6$ and the following

⁶ Notice that this implies that z lies in the positive real axis.

redefinition of the dependent variable $C[z] = z^{\frac{-i\omega}{2}}(1-z)^{\frac{1-\lambda}{2}}G[z]$ one gets a new form for the original differential equation:

$$D_4 G[z] = 0. \quad (4.54)$$

The differential operator D_4 , whose precise form is given in (C.13), has three regular singularities at 0, 1 and ∞ with the following $4 \times 3 = 12$ characteristic exponents:

$$\begin{aligned} \alpha_0^I &= (0, i\omega) & \alpha_1^I &= (0, \lambda) & \alpha_\infty &= (\delta_+^+, \delta_+^+) \\ \alpha_0^{II} &= (1, 1 + i\omega) & \alpha_1^{II} &= (1, 1 + \lambda) & \tilde{\alpha}_\infty &= (\delta_-^-, \delta_+^-), \end{aligned}$$

where:

$$\begin{aligned} \delta_+^+ &= \frac{1-\lambda}{2} + \delta_0^+(Q_2), & \delta_-^+ &= \frac{1-2i\omega-\lambda}{2} - \delta_0^+(Q_2), \\ \delta_+^- &= \frac{1-\lambda}{2} + \delta_0^-(Q_2), & \delta_-^- &= \frac{1-2i\omega-\lambda}{2} - \delta_0^-(Q_2), \end{aligned} \quad (4.55)$$

and:

$$\delta_0^\pm(\mu_3) = \frac{-3 \pm \sqrt{9 - 36i\mu_3(\omega + k) + 12\mu_3^2(\lambda^2 - 1)}}{12\mu_3}. \quad (4.56)$$

Notice that δ_0^+ is regular in the limit of vanishing μ_3 whereas δ_0^- is not.

For a Fuchsian differential equation of order n with m regular singular points the sum of characteristic exponents is always $(m-2) \times \frac{n(n-1)}{2}$ [65]. It is easy to check that in our case $n = 4$, $m = 3$ the sum of characteristic exponents is indeed 6. An interesting case is when $n = 2$ and $m = 3$ in that case one has $m \times n = 6$ characteristic exponents whose sum equals 1. Conversely, it is a theorem that any set of 6 numbers adding up to 1 defines a unique Fuchsian operator of order $n = 2$ with $m = 3$ regular singular points. It is also a theorem that such a sextuplet of roots defines a subspace of solutions that carry an irreducible representation of the monodromy group of D_n and hence a factor D_2 [65]. Namely:

$$D_n = D_{n-2}^L D_2^R, \quad (4.57)$$

and D_{n-2}^L is also Fuchsian and the L and R denote the left and right operator, respectively, in the factorization.

Before proceeding, let us review some facts that will be used in the following [65, 66]. The most general form of a Fuchsian differential operator D_2 once the position of the regular singular points are fixed at 0, 1, ∞ and a pair of characteristic exponents is fixed to zero, is:

$$D_2 \equiv y(y-1) \frac{d^2}{dy^2} + ((a+b+1)y - c) \frac{d}{dy} + ab. \quad (4.58)$$

The characteristic exponents are:

$$\alpha_0 = (0, 1 - c), \quad \alpha_1 = (0, c - a - b), \quad \alpha_\infty = (a, b). \quad (4.59)$$

The kernel of D_2 is generated by the linearly independent functions:

$$\begin{aligned} u_1(a, b, c|z) &\equiv {}_2F_1(a, b, c|z), \\ z^{1-c}u_2(a, b, c|z) &\equiv z^{1-c}{}_2F_1(a+1-c, b+1-c, 2-c|z), \end{aligned} \quad (4.60)$$

which are eigenstates of the monodromy action at $z = 0$. The second solution is independent only when c is not in \mathbb{Z} . The monodromy eigenstates at $z = 1$ are:

$$\begin{aligned} \tilde{u}_1(a, b, c|z) &\equiv {}_2F_1(a, b, 1+a+b-c|1-z), \\ (1-z)^{c-a-b}\tilde{u}_2(a, b, c|z) &\equiv (1-z)^{c-a-b}{}_2F_1(c-a, c-b, 1+c-a-b|1-z). \end{aligned} \quad (4.61)$$

When $c - a - b$ is not in \mathbb{Z} . In a while we will see that $c - a - b = \lambda$.

Our operator D_4 does have the properties mentioned in the paragraph before (4.57). In fact the set of characteristic exponents:

$$\begin{aligned} (\alpha_0^I, \alpha_1^I, \alpha_\infty^I), \\ (\alpha_0^I, \alpha_1^I, \tilde{\alpha}_\infty^I), \end{aligned} \quad (4.62)$$

adds up to 1, and hence defines the second order Fuchsian operators:

$$\begin{aligned} D_2^R &: a = \delta_+^+(\mu_3), \quad b = \delta_-^+(\mu_3), \quad c = 1 - i\omega, \\ \tilde{D}_2^R &: a = \delta_+^-(\mu_3), \quad b = \delta_-^-(\mu_3), \quad c = 1 - i\omega. \end{aligned} \quad (4.63)$$

As a result D_4 has two independent factorizations:

$$D_4 = D_2^L D_2^R \text{ and } D_4 = \tilde{D}_2^L \tilde{D}_2^R, \quad (4.64)$$

as one can check explicitly. Consequently we have:

$$\ker D_4 = \ker D_2^R \oplus \ker \tilde{D}_2^R, \quad (4.65)$$

where $\ker D_2^R$ is given by the hypergeometric functions u_1 and u_2 given in (4.60), with the parameters a , b and c defined in (4.63). This proves that the fluctuation equation in the background $\mu_3 \neq 0$ is solved in terms of four linearly independent hypergeometric functions, which, from now on we refer to as “building blocks”.

One can explicitly verify this factorization pattern for the next background, with $\mu_3, \mu_5 \neq 0$. In this case $s_{Max} = 3$ and the corresponding differential operator D_8 , has order 8, and is again Fuchsian with 3 regular singularities in the z coordinate system previously defined (we always place them at 0, 1 and ∞). The characteristic exponents are:

$$\begin{array}{lll} \alpha_0^I = (0, i\omega) & \alpha_1^I = (0, \lambda) & \alpha_\infty^I = (\delta_-^{++}, \delta_+^{++}) \\ \alpha_0^{II} = (1, 1 + i\omega) & \alpha_1^{II} = (1, 1 + \lambda) & \alpha_\infty^{II} = (\delta_-^{+-}, \delta_+^{+-}) \\ \alpha_0^{III} = (2, 2 + i\omega) & \alpha_1^{III} = (2, 2 + \lambda) & \alpha_\infty^{III} = (\delta_-^{-+}, \delta_+^{-+}) \\ \alpha_0^{IV} = (3, 3 + i\omega) & \alpha_1^{IV} = (3, 3 + \lambda) & \alpha_\infty^{IV} = (\delta_-^{--}, \delta_+^{--}), \end{array}$$

where for each of the couples of exponents α_∞ the following property holds: $\delta_+^{\pm\pm}(\mu_3, \mu_5) + \delta_-^{\pm\pm}(\mu_3, \mu_5) = 1 - i\omega - \lambda$. As a consequence there are four triads of characteristic exponents whose sums equals 1 :

$$\begin{aligned} &(\alpha_0^I, \alpha_1^I, \alpha_\infty^I), \quad (\alpha_0^I, \alpha_1^I, \alpha_\infty^{II}), \\ &(\alpha_0^I, \alpha_1^I, \alpha_\infty^{III}), \quad (\alpha_0^I, \alpha_1^I, \alpha_\infty^{IV}). \end{aligned} \quad (4.66)$$

Each of them defines a second order ‘‘Hypergeometric operator’’ as in (4.63):

$$D_2^{IR}, D_2^{IIR}, D_2^{IIIR} \text{ and } D_2^{IVR}$$

such that

$$\ker D_8 = \ker D_2^{IR} \oplus \ker D_2^{IIR} \oplus \ker D_2^{IIIR} \oplus \ker D_2^{IVR}.$$

In fact there is a simple way to prove that the above pattern generalizes, showing that the solutions of our high order differential equations can be expressed in terms of ordinary hypergeometric functions, for all of the representatives in (4.14). The point is to use the fact that the Fourier components $C(\omega, k)$ of the full master field $C(t, x)$ defined by the arbitrary polynomial P_ϕ and \bar{P}_ϕ , are eigenstates of the operators in the right hand side of:

$$\begin{aligned} \partial_t &= \frac{-a_\rho \star_L + a_{-\rho} \star_R}{2}, \\ \partial_\phi &= -\left(\frac{a_\rho}{2} + P_\phi(a_\rho)\right) \star_L - \left(\frac{a_{-\rho}}{2} - \bar{P}_\phi(a_{-\rho})\right) \star_R, \end{aligned} \quad (4.67)$$

with eigenvalues $(i\omega, ik)$ respectively. The same can be said of the trace component $C_0^1(\omega, k)$ but in this case, the left and right multiplication are equivalent by cyclic property of the trace. As the operators on the right hand side of (4.67) are polynomials in $a_{\pm\rho}$, they share the eigenvectors with the latter. But as we pointed out around (4.46):

$$\begin{aligned} i(\omega' + k') C_{BTZ}(\omega', k') &= -a_\rho \star_L C_{BTZ}(\omega', k'), \\ i(k' - \omega') C_{BTZ}(\omega', k') &= -a_{-\rho} \star_R C_{BTZ}(\omega', k'), \end{aligned} \quad (4.68)$$

where C_{BTZ} is the master field for the $BTZ_{M=-1}$ connection. So from (4.67) and (4.68) it follows that:

$$C_0^1(\omega, k) = C_{0BTZ}^1(\omega', k'), \quad (4.69)$$

where (ω', k') are any of the roots of the algebraic equations:

$$\begin{aligned} i\omega &= i\omega', \\ ik &= ik' - (P_\phi(-i(\omega' + k')) - \bar{P}_\phi(-i(k' - \omega'))). \end{aligned} \quad (4.70)$$

Relations (4.69) imply that the differential equation for C_0^1 in the class of ansätze (4.14) is always integrable in terms of hypergeometric functions ${}_2F_1$. The number of linearly independent modes being given by twice the order of the algebraic equations (4.70), which can be checked to be, $2(s_{Max} + 1)$. Here $s_{Max} + 1$ coincides with order of the polynomial equation (4.70) for k' in terms of (ω, k) .

Summarizing, the most general solution for fluctuations in (4.14) is:

$$C_0^1(\omega, k) = \sum_r e^{i(\omega t + k\phi)} (1 - z)^{\frac{1-\lambda}{2}} \left(c_r^{in} z^{-\frac{i\omega}{2}} u_1(a_r, b_r, 1 - i\omega, z) + c_r^{out} z^{\frac{i\omega}{2}} u_2(a_r, b_r, 1 - i\omega, z) \right),$$

$$a_r \equiv \frac{i(k'_r - \omega) + 1 - \lambda}{2}, \quad b_r \equiv \frac{-i(k'_r + \omega) + 1 - \lambda}{2}, \quad (4.71)$$

where k'_r are the roots of (4.70) and $r = 1, \dots, 2(s_{Max} + 1)$.

For later reference we write down (4.71) in terms of monodromy eigenstates at the boundary $z = 1$:

$$C_0^1(\omega, k) = \sum_r e^{i(\omega t + k\phi)} z^{-\frac{i\omega}{2}} (1 - z)^{\frac{1-\lambda}{2}} \left(\tilde{c}_r^1 \tilde{u}_1(a_r, b_r, 1 - i\omega; z) + \tilde{c}_r^2 (1 - z)^\lambda \tilde{u}_2(a_r, b_r, 1 - i\omega; z) \right). \quad (4.72)$$

As a check, let us reproduce the first result of this section by using this method. For the case $\mu_3 \neq 0$ the equation for k'_r are:

$$ik = ik'_r - \mu_3 \left(-(\omega + k'_r)^2 + \frac{1 - \lambda^2}{3} \right), \quad (4.73)$$

whose solutions are :

$$ik'_\pm = -i\omega - \delta_0^\pm(\mu_3). \quad (4.74)$$

This coincides with the solution one obtains from (4.63), as can be seen using the definitions in the second line of (4.71). We note that only k'_+ is smooth in the BTZ limit μ_3 to zero.

In conclusion, we mention the fact that the boundary conditions for the most general fluctuation (4.71) at the horizon and boundary, $z = 0$ and $z = 1$, respectively, are not affected by the fact that the background tensor fields $g^{(n)}$, defined as (4.2), and starting by the metric $n = 2$, do not satisfy the original $BTZ_{M=-1}$ asymptotics.

4.3 QNM and bulk to boundary 2-point functions

As anticipated, in this subsection we will further argue that the connections (4.14) describe a class of Black Hole configurations. We will do so by showing the presence

of Quasi Normal Modes. We will compute their spectrum for any representative in (4.14) and, in particular, more explicitly for the simplest cases discussed in the previous section.

We start by recalling the conditions for QNM for AdS Black Holes[67]: they behave like ingoing waves at the horizon, $z = 0$ and as subleading modes at the boundary $z = 1$. In the language employed before, the QNM conditions reduce to ask for solutions with indicial roots $\alpha_0 = 0$ at the horizon $z = 0$, and $\alpha_1 = \lambda$ at the boundary $z = 1$. In this section we are considering the region $0 < \lambda < 1$ so that $(1 - z)^{\frac{(1-\lambda)}{2}}$ is the leading behaviour near the boundary. In terms of the most general solution (4.71), the ingoing wave condition reads: $c_r^{out} = 0$. The subleading behaviour requirement implies the quantization conditions⁷.

$$\omega \pm k'_r + i(1 + 2n + \lambda) = 0, \quad r = 0, \dots, 2(s_{Max} + 1), \quad (4.75)$$

where n is an arbitrary and positive integer.

We should elaborate about the smoothness of the QNM at the horizon. In the Eddington-Finkelstein coordinates $v = t - \frac{T}{2} \log(\rho) + \dots$ and $\rho^* = \frac{4}{T^2} \rho^2 + \dots$, see (4.15) the incoming waves, namely the c_r^{in} modes, behave as plane waves $e^{I_{\omega v}}$, at leading order in the near to horizon expansion. In contrast, the c_r^{out} modes are not C^∞ as they look like $e^{i\omega v} (\rho^{*i\omega})$. In other words, the requirement of incoming waves at the horizon amounts to have a smooth solution at the horizon[67].

In our example $\mu_3 \neq 0$, $s_{Max} = 1$, there are 2×2 branches in the quantization conditions (4.75). The associated branches of QNM being:

$$\begin{aligned} \omega_n^0 &= -k - i \left(1 + 2n + \lambda - \frac{2\mu_3}{3} (1 + (1 + 2\lambda)(1 + \lambda) - \lambda^2 + 6n(1 + \lambda) + 6n^2) \right), \\ \omega_n^\pm &= -\frac{1}{2}i(1 + 2n + \lambda) + \delta^\pm(n, \mu_3), \end{aligned} \quad (4.76)$$

where:

$$\delta^\pm(n, \mu_3) = \frac{-i \pm \sqrt{-1 + 8(1 + 2ik + 2n + \lambda)\mu_3 - \frac{16(\lambda^2 - 1)\mu_3^2}{3}}}{8\mu_3}. \quad (4.77)$$

Before going on, let us briefly mention some relevant issues about the stability of the branches (4.76). It is not hard to see that for large enough values of $k \in \mathbb{R}$ at least one of the branches ω_n^\pm will exhibit a finite number of undamped modes, namely modes with positive imaginary parts. However for a fixed value of k and μ_3 the UV modes ($n \gg 1, k, \mu_3$) will go like $\omega_n^\pm \sim -in$ and hence will be stable. The branch ω_n^0 is stable for $\mu_3 < 0$. Finally notice also that (ω_n^0, ω_n^+) become the left and right moving branches of the BTZ $_{M=-1}$ case, in the limit of vanishing μ_3 , whereas ω_n^- is

⁷ We have the identity ${}_2F_1[a, b, c, z] = \frac{\Gamma[c]\Gamma[a+b-c]}{\Gamma[c-b]\Gamma[c-a]} {}_2F_1[a, b, a+b-c+1, 1-z] + (1-z)^{c-a-b} \frac{\Gamma[c]\Gamma[c-a-b]}{\Gamma[b]\Gamma[a]} {}_2F_1[c-a, c-b, c-a-b+1, 1-z]$ [66]. The quantization condition (4.75) is equivalent to $c-a = -n$ and $c-b = -n$ respectively. These choices guarantees that the first term on the rhs of the previous identity vanishes. This is indeed the term that carries the leading behaviour of the field at the boundary.

not analytic in that limit.

We have $2 \times 2(s_{Max} + 1)$ independent solutions $(c^{in}, c^{out})_r$ in (4.71). Each block r represents an independent degree of freedom and a general fluctuation in the background (4.14) can be re-constructed as a combination of them. So, for the moment we restrict our analysis to a given sector, let us say the block r .

In order to define the bulk to boundary 2-point function we set $\tilde{c}_r^2 = 0$ in (4.72), corresponding to the solution with the leading behaviour $(1-z)^{\frac{1-\lambda}{2}}$ at the boundary. We will further fix $\tilde{c}_r^1 = 1$, to guarantee independence on ω and k of the leading term in the expansion of the solution near the boundary, in such a way that its Fourier transform becomes proportional to $\delta^{(2)}(t, \phi)$ at the boundary, which is the usual UV boundary condition in coordinate space. As a result, in Fourier space, the bulk to boundary 2-point function of the block of solutions r is given by:

$$G_r^{(2)}(\omega, k, z) \equiv \tilde{u}_1(a_r, b_r, 1 - i\omega; 1 - z). \quad (4.78)$$

After Fourier transforming back in (t, ϕ) space and using the ρ coordinate:

$$G_r^{(2)}(t, \phi, \rho) = J_r(-i\partial_t, -i\partial_\phi) \left(G_{BTZ}^{(2)}(t, \phi; \rho) + \delta G_r^{(2)}(t, \phi, \rho) \right). \quad (4.79)$$

We stress that (4.79) obeys the boundary condition:

$$G_r^{(2)}(t, \phi, \rho) \rightarrow \delta^{(2)}(t, \phi), \text{ when } \rho \rightarrow \infty. \quad (4.80)$$

The quantity:

$$J_r(\omega, k) \equiv \frac{1}{\frac{\partial k'_r(\omega, k)}{\partial k}} e^{i(k - k'_r(\omega, k))\phi},$$

is the product of the Jacobian from the change of variables from k to k'_r times an exponential contribution. For our specific case:

$$J_r(\omega, k) = (1 + 2i\mu_3\delta_0^\pm(\omega, k)) e^{i(k - k'_r(\omega, k))\phi}. \quad (4.81)$$

The quantity:

$$G_{BTZ}^{(2)}(t, \phi, \rho) = -\frac{\lambda}{\pi} \left(\frac{e^{-\rho}}{e^{-2\rho} \cosh x_+ \cosh x_- + \sinh x_+ \sinh x_-} \right)^{1-\lambda}, \quad (4.82)$$

is the bulk to boundary 2-point function for $BTZ_{M=-1}$. Notice that (4.82) is smooth in the near-horizon expansion as its leading contribution is independent of t . We note that the contributions coming from $G_{BTZ}^{(2)}$ to (4.79) are also smooth at the horizon provided the Taylor expansion of $J_r(w, k)$ around $(\omega, k) = 0$ starts with a constant or an integer power of k . This is always the case, as one can infer from (4.70) that $J_r = 1 + O(\mu_3)$, as in the particular case (4.81).

Finally $\delta G_r^{(2)}$ is a contribution that comes from the deformation of the contour of integration that follows from the change $k \rightarrow k'_r$. The change of variable from k

to $k'_r(\omega, k)$ deforms the real line \mathbb{R} to a contour $C_{r,\omega} \equiv k'_r(\mathbb{R}, \omega)$. Integration over the contours $k' \in \mathbb{R}$ and $k'_r \in C_{r,\omega}$ (followed by integration over $\omega \in \mathbb{R}$) of the integrand

$$e^{ik'_r\phi + i\omega t} \tilde{u}_1(a_r, b_r, 1 - i\omega; 1 - z),$$

differ by the quantity $\delta G_r^{(2)}(t, \phi, z)$. This quantity can be obtained imposing the condition (4.80). In Fourier space (ω, k'_r) It reads:

$$\delta G_r^{(2)}(\omega, k'_r, z) = \left(\frac{\partial k'_r}{\partial k} - 1 \right) \tilde{u}_1(a_r, b_r, 1 - i\omega; 1 - z).^8 \quad (4.83)$$

Finally, (4.79) takes the form:

$$G_r^{(2)}(t, \phi, \rho) = e^{-\left(ik'_r(-i\partial_t, -i\partial_\phi) - \partial_\phi\right)\phi} G_{BTZ}^{(2)}(t, \phi, \rho).^9 \quad (4.84)$$

Also as already said (4.84) is smooth at the horizon, as its leading behavior is independent on t .

Notice that periodicity under $t \rightarrow t + 2\pi i$ is preserved by all building blocks (4.84). The preservation of thermal periodicity comes after imposing the BTZ holonomy condition on (4.14). It is a global statement in the sense that is determined by the exponentiation properties of the algebra. Namely the gauge group elements generating the family (4.14) with $\nu_0 = \frac{1}{2}$, $\nu_{i>0} = 0$:

$$\begin{aligned} g &= e^{-\rho V_0^2} e^{-\frac{a}{2}t - \left(\frac{a}{2} + P_\phi(a)\right)\phi}, \\ \bar{g} &= e^{\rho V_0^2} e^{-\frac{a}{2}t + \left(\frac{a}{2} - \bar{P}_\phi(a)\right)\phi}, \end{aligned} \quad (4.85)$$

are thermal periodic due to the fact $i\pi a$ exponentiates to the center of the group whose Lie algebra is $\mathfrak{hs}(\lambda)$ [62].

4.4 Making Contact with other Relevant Backgrounds

In this section we perform the gauge transformations (4.24) and (4.26) taking our backgrounds to the GK (BHPTT) ones. As already said, the backgrounds to be transformed have critical exponent $z < 1$. Here we will focus in performing gauge transformations (4.24) and (4.26) on the scalar fluctuations for $\bar{\mu}_3 = -\mu_3 \neq 0$ and we will explicitly verify that they solve the equation of motion for matter fluctuations in the GK (BHPTT) backgrounds. The analysis will be done perturbatively, to first order in a μ_3 expansion.

To this purpose we introduce the series expansion:

$$C = \sum_{i=0}^{\infty} \mu_3^i C^{(i)}, \quad (4.86)$$

⁸ Notice that the quantity $\delta G_r^{(2)}(\omega, k'_r, z)$ (as $G_{BTZ}^{(2)}(\omega, k'_r, z)$) is in the kernel of the BTZ Klein-Gordon operator $D_2(\omega, k'_r, z)$.

⁹ We note that the ϕ in the exponential (4.84) is located to the right of the derivatives.

for the master field in equations (4.28) with the connections (A, \bar{A}) given by (4.23), (4.25) and (4.27). Taking the μ_3^i component of (4.28):

$$(d + \overset{(0)}{\mathcal{A}} \star_L - \overset{(0)}{\bar{\mathcal{A}}} \star_R) \overset{(i)}{C} = - \sum_{j=1}^i \overset{(j)}{\mathcal{A}} \star_L - \overset{(j)}{\bar{\mathcal{A}}} \star_R \overset{(i-j)}{C}, \quad i = 0, \dots, \infty, \quad (4.87)$$

where $\overset{(j)}{\mathcal{A}}$ is the coefficient of μ_3^j in the Taylor expansion of \mathcal{A} about $\mu_3 = 0$. Notice that if $\overset{(i)}{C}$ is a particular solution of (4.87), then $\overset{(i)}{C} + \text{constant } \overset{(0)}{C}$ is also a solution. This is in fact the maximal freedom in defining $\overset{(i)}{C}$ and we obtain the following type of equations:

$$\begin{aligned} D_2 \overset{(0)}{C}_0^1 &= 0, \quad i = 0, \\ D_2 \overset{(i)}{C}_0^1 &= \overset{(i)}{D} \left(\overset{(0)}{C}_0^1, \dots, \overset{(i-1)}{C}_0^1 \right), \quad i = 1, \dots, \infty, \end{aligned} \quad (4.88)$$

after applying the folding method. The differential operator D_2 is the BTZ Klein-Gordon operator (4.49) and $\overset{(i)}{D}$ is a linear differential operator in ρ that we shall find out explicitly when analyzing up to first order in μ_3 .

Let us write the connections (4.14) with $\mu_3 = -\bar{\mu}_3 \neq 0$ as:

$$\mathcal{A}_{ours} = \overset{(0)}{\mathcal{A}} + \mu_3 \overset{(1)}{\mathcal{A}}_{ours}, \quad \bar{\mathcal{A}}_{ours} = \overset{(0)}{\bar{\mathcal{A}}} + \mu_3 \overset{(1)}{\bar{\mathcal{A}}}_{ours}. \quad (4.89)$$

The full answer C_{0ours}^1 is defined as the building block r in (4.71) with k'_r , given by the root (C.5) of equation (C.4) which is the analytic solution in the limit μ_3 to zero. By using the folding method one can check until arbitrary order in i that (4.88) works for the expansion coefficients $\overset{(i)}{C}_{ours}$. Here we restrict to the $i = 1$:

$$D_2 \overset{(1)}{C}_{0ours}^1 = \overset{(1)}{D}_{ours} \overset{(0)}{C}_0^1, \quad (4.90)$$

where:

$$\overset{(1)}{D}_{ours} = \frac{16ike^{2\rho} \left(\frac{1}{3}(\lambda^2 - 1) + k^2 + w^2 \right)}{(e^{2\rho} + 1)^2}. \quad (4.91)$$

Let us solve (4.90). We can expand in series the solution for C_{0ours}^1 (4.71), but we will use gauge covariance instead. From the use of the transformation laws:

$$\begin{aligned} \mathcal{A}_{ours} &= e^{\Lambda_{ours}} \mathcal{A} e^{-\Lambda_{ours}} + e^{\Lambda_{ours}} d e^{-\Lambda_{ours}}, \\ \bar{\mathcal{A}}_{ours} &= e^{\bar{\Lambda}_{ours}} \bar{\mathcal{A}} e^{-\bar{\Lambda}_{ours}} + e^{\bar{\Lambda}_{ours}} d e^{-\bar{\Lambda}_{ours}}, \end{aligned} \quad (4.92)$$

at linear order, with:

$$\Lambda_{ours} = -\phi P_\phi(a_\rho), \quad \bar{\Lambda}_{ours} = -\phi \bar{P}_\phi(a_{-\rho}). \quad (4.93)$$

And $C_{0\text{ours}}^1 = \left((e^{\Lambda_{\text{ours}} - \bar{\Lambda}_{\text{ours}}}) C_0^1 \right) \Big|_{V_0^1}$, for the case $\mu_3 = -\mu_3 \neq 0$ and in Fourier space gives:

$$\begin{aligned} C_{0\text{ours}}^{(1)} &= -i\partial_k \left((a_\rho^2 + a_{-\rho}^2 - \text{trace}) C_0^{(0)} \right) \Big|_{V_0^1} \\ &= -i \left(\frac{2}{3}(1 - \lambda^2) - 2(k^2 + w^2) \right) \partial_k C_0^{(0)} + \dots, \end{aligned} \quad (4.94)$$

where the \dots in (4.94) stand for terms that are proportional to $C_0^{(0)}$ and hence are in the kernel of D_2 .

To check that (4.94) is solution of (4.90) it is enough to check that:

$$\left[i \left(\frac{2}{3}(1 - \lambda^2) - 2(k^2 + w^2) \right) \partial_k, D_2 \right] = D_{\text{ours}}^{(1)}, \quad (4.95)$$

by using (C.12) or to notice that (4.94) coincides with the first order coefficient in the Taylor expansion around $\mu_3 = 0$ of the corresponding solution $C_{0\text{ours}}^1$ which is given by $(\frac{\partial k'}{\partial \mu_3} \partial_{k'} C_{0\text{ours}}^1)_{\mu_3=0} = \frac{\partial k'}{\partial \mu_3} \Big|_{\mu_3=0} \partial_k C_0^{(0)}$.

Next we truncate the GK background at first order in μ_3 and after following the procedure we can explicitly show again that the form (4.88) holds until $i = 1$ ¹⁰. Here we just present the $i = 1$ equation:

$$D_2 C_{0GK}^{(1)} = D_{GK}^{(1)} C_0^{(0)}. \quad (4.96)$$

The expression for $D_{GK}^{(1)}$ is given in (C.14). We should stress again that (4.96) refers only to fluctuations over the GK ansatz that are analytic when μ_3 goes to zero. Finally we check explicitly that the transformed fluctuation:

$$\begin{aligned} C_{0GK}^{(1)} &= C_{0\text{ours}}^{(1)} + \left((\Lambda_{GK}^{(1)} - \bar{\Lambda}_{GK}^{(1)}) C_0^{(0)} \right) \Big|_{V_0^1} \\ &= C_{0\text{ours}}^{(1)} - \frac{ik(3e^{2\rho} + 5)}{3(e^{2\rho} + 1)^2} \left((e^{2\rho} - 1) C_0^{(0)} - (e^{2\rho} + 1) \partial_\rho C_0^{(0)} \right), \end{aligned} \quad (4.97)$$

solves (4.96) after using (4.90) and the $i = 0$ equation in (4.88). We have then reproduced the result of [68, 69], by starting from our ansatz.

¹⁰ We checked it up to $i = 2$, when the GK background is truncated at second order in μ_3 .

4.5 Final Remarks

We have presented a family of connections constructed out of arbitrary polynomial combinations of the $\text{BTZ}_{M=-1}$ connection in $\mathfrak{hs}(\lambda) \times \mathfrak{hs}(\lambda)$ 3D CS theory. Their space time tensor fields present smooth horizons. The system of higher order differential equations of motion for matter fluctuations can be solved in terms of hypergeometric functions related to the solutions in the BTZ background. This allows to solve explicitly for Quasi Normal Modes and 2-point functions. As a check, we have made contact with other backgrounds studied in the literature. Among the open problems that this work leaves unanswered, we mention the following ones. The first regards the understanding of which (higher spin ?) charges are carried by our backgrounds, or, more generally what is the asymptotic symmetry algebra associated to them. Recent progresses on this problem for BH backgrounds in the $\mathfrak{sl}(3)$ CS theory, may allow to get an answer for our cases. Secondly, one would like to use the results we found for the matter fluctuations to solve for more general backgrounds by using appropriate gauge transformations (either “proper” or “improper”) carrying our backgrounds to these. Unfortunately, a perturbative analysis along the lines discussed in this paper seems to be unavoidably beset by singularities at the horizon $\rho = 0$. It would be interesting to know whether this is an artifact of the perturbative expansion and if a full non perturbative analysis would be free of such singularities. This would allow to study QNM virtually for any BH background. We owe a more detailed study of the properties of the differential operators governing the propagation of matter in our backgrounds. Perhaps this study could shed some light on the specific geometrical properties that drive matter propagation in backgrounds with higher spins [62]. Finally, we stress that the same approach we followed to show the factorization property, can be implemented for a family of backgrounds constructed out of polynomials of more general highest weight connections. We hope to come back to some of these issues in the near future.

5. SUPERSYMMETRIC LOCALIZATION

After the seminal computations [42, 70] supersymmetric localization has been lately a recurrent technique to compute certain observables, principally by placing supersymmetrically theories on compact manifolds, to mention some works [71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83]. Less studied on manifolds with boundaries [84, 85, 86] and in supergravity in [87, 88, 89, 90]. Some extensive reviews and early literature are [41, 91, 92, 93, 94]. Now we will give, with an approach rather heuristic the general idea, we will be making a mere formal treatment of the path integrals.

5.1 General Idea

Let \mathcal{Q} be a non-anomalous supersymmetry acting on the fields $\Phi \in \mathcal{F}^1$ of your theory (a Grassmann-odd symmetry) which has an action S , and is such that its algebra closes to the symmetries of your theory, *i.e*

$$\mathcal{Q}^2 = B, \quad (5.1)$$

where B is a bosonic symmetry of your theory, which can be a linear combination of space-time symmetries, global internal symmetries, and gauge symmetries. Let us denote by G the group generated by \mathcal{Q} . In a supersymmetric theory we have a special set of observables, the ones which are annihilated by a given supersymmetry and they are dubbed BPS operators, we will denote as $\mathcal{O} \in \mathcal{F}_{\mathcal{Q}}$ a generic operator close under \mathcal{Q} .

The first localization argument for supersymmetric theories is the following by Witten [92], if the action of G is free one can write for the expectation value for \mathcal{O} as:

$$\langle \mathcal{O} \rangle = \int_{\mathcal{F}} [D\Phi] \mathcal{O} e^{-S} = \int_G \int_{\mathcal{F}/G} [D\Phi] \mathcal{O} e^{-S}, \quad (5.2)$$

As G is generated by \mathcal{Q} , the integral over G will be denoted by an integral over a collective Grassmann variable θ , since the action of \mathcal{Q} is free, there is no θ dependence in the integrand and therefore the integral has to vanish. Luckily, the action of the of G is not free, and we have in the space of fields the BPS observables. Let $\mathcal{F}_{\mathcal{Q}} \subset \mathcal{F}$ be the space of BPS field configurations and \mathcal{F}' a neighborhood of $\mathcal{F}_{\mathcal{Q}}$ so we can

¹ We denote collectively the fields of theory as Φ , but also might mean a generic supermultiplet, instead Ψ_{Φ} is a generic fermion corresponding to a particular multiplet Φ , \mathcal{F} is the space of fields of the theory, and $\mathcal{F}_{\mathcal{Q}}$ is the space of \mathcal{Q} -observables

write:

$$\langle \mathcal{O} \rangle = \int_{\mathcal{F}} [D\Phi] \mathcal{O} e^{-S} = \int_{\mathcal{F}'} [D\Phi] \mathcal{O} e^{-S} + \int_{\mathcal{F}'^\perp} [D\Phi] \mathcal{O} e^{-S}, \quad (5.3)$$

The action of G on the complement of \mathcal{F}' , \mathcal{F}'^\perp is free, therefore the second integral in (5.3) is zero. The neighborhood can be taken arbitrary small and therefore the integral localizes to \mathcal{F}_Q .

But this is not all, suppose we can find and Q -exact deformation $Q\mathcal{V}$ of the action parametrize by t , such that $B\mathcal{V} = 0$, *i.e*

$$\langle \mathcal{O} \rangle_t = \int_{\mathcal{F}} [D\Phi] \mathcal{O} e^{-S+t \int_{\mathcal{M}} Q\mathcal{V}}, \quad (5.4)$$

By taking derivative respect t we get,

$$\frac{d\langle \mathcal{O} \rangle_t}{dt} = \int_{\mathcal{F}} Q([D\Phi] \mathcal{O} \mathcal{V} e^{-S+t \int_{\mathcal{M}} Q\mathcal{V}}), \quad (5.5)$$

Where we have used the fact that Q is non-anomalous and that $Q\mathcal{V}$ and S are Q -close, if we assume that the functional integral above does not have any boundary contribution, so the integrand decay fast enough. The total variation integral in (5.5) vanishes, and,

$$\frac{d\langle \mathcal{O} \rangle_t}{dt} = 0, \quad (5.6)$$

this means that the actual value that t takes is not important, and the original observable

$$\langle \mathcal{O} \rangle_{t=0} = \langle \mathcal{O} \rangle_t \quad \forall t \in \mathbb{R}^+, \quad (5.7)$$

Indeed what it is done usually is to construct $Q\mathcal{V}$ which is positive semi-definite, and by taking the limiting value $t \rightarrow \infty$ the integrand will be controlled by the points where $Q\mathcal{V} = 0$, and then the path integral localizes to fixed point set of $Q\mathcal{V}$ which we denote by $\mathcal{F}_{\mathcal{V}}$, this is why the Q -deformation is also called localization action $S_{loc} = Q\mathcal{V}$. By combining the two arguments of localization the path integral will localize to:

$$\mathcal{F}_{loc} = \mathcal{F}_Q \cap \mathcal{F}_{\mathcal{V}} \quad \text{and} \quad \mathcal{F}_{\mathcal{V}} \subset \mathcal{F}_Q, \quad (5.8)$$

In practice what happens many times is that $\mathcal{F}_{\mathcal{V}} = \mathcal{F}_Q$ and this is done by picking the the following deformation, which is by now standard in the literature,

$$\mathcal{V} = \sum_{\Psi_\Phi} (Q\Psi_\Phi)^\dagger Q\Psi_\Phi, \quad (5.9)$$

where the trace over gauge, internal and fermion indices is understood, the symbol \dagger is a proper definition of complex conjugation such that the deformation is positive definite. It is clear now from (5.9) that the fixed point set of $Q\mathcal{V}$ is nothing but:

$$Q\Psi_\Phi = 0 \quad \forall \Psi_\Phi, \quad (5.10)$$

this is nothing but Q -BPS condition.

5.2 Saddle Point “Approximation”

To evaluate the path integral of the previous section it is convenient to expand the fields in fluctuations around the fixed points Φ_0 of the localization action, one might think the parameter t as making the work of the Planck constant.

$$\Phi = \Phi_0 + \frac{\Phi'}{\sqrt{t}}, \quad (5.11)$$

The semi-classical approximation of the integrand in (5.4) is given by the zeroth order in t , *i.e* up quadratic order in the fluctuations

$$S[\Phi_0] + \frac{\delta^2 S_{loc}}{\delta \Phi^2} \Big|_{\Phi=\Phi_0} \Phi'^2 + \dots, \quad (5.12)$$

But when we take $t \rightarrow \infty$, the higher order contributions vanish and the above result is exact, then we have (5.4) reduces to

$$\begin{aligned} \langle \mathcal{O} \rangle &= \int_{\mathcal{F}_{loc}} [d\Phi_0] \mathcal{O}[\Phi_0] e^{-S[\Phi_0]} \int [d\Phi'] e^{-\frac{\delta^2 S_{loc}[\Phi_0]}{\delta \Phi^2} \Phi'^2} \\ &= \int_{\mathcal{F}_{loc}} [d\Phi_0] \mathcal{O}[\Phi_0] \frac{e^{-S[\Phi_0]}}{SDet[\frac{\delta^2 S_{loc}[\Phi_0]}{\delta^2 \Phi}]}, \end{aligned} \quad (5.13)$$

Where we wrote schematically the integrations over bosonic and fermionic fluctuations as $SDet[\frac{\delta^2 S_{loc}[\Phi_0]}{\delta^2 \Phi}]$.

5.3 Rigid Supersymmetry on Curved Backgrounds

The main ingredient of the previous sections is obviously supersymmetry, the calculations that have been performed placed the theories typically in compact manifolds. This has the nice property of regularizing the IR divergences of the theory, confining the dynamics at finite volume.

To construct supersymmetry in such spaces there have been several approaches. One can pragmatically say that is enough to find supersymmetry by covariantizing and adding recursively corrections on the typical size of the space. Approaches like this were carried out in [77, 78]. What one needs at the end is that the supersymmetry algebra closes and a sensible Lagrangian. Another possibility if the manifold is conformally flat like in [70] is to begin with a supersymmetric theory in higher dimensions and then dimensionally reducing, and use the conformal properties to add the proper pieces.

There is also a more powerful approach derived initially in [95] for $\mathcal{N} = 1$ 4D theories (or $\mathcal{N} = 2$ in 3D), and consist basically of taking the rigid limit of supergravity theory. The principal ingredient is to keep always the auxiliary fields in the gravity multiplet as arbitrary background fields, these fields will be crucial to generalize the killing spinor equations. As described in [95, 96, 97] one takes $\mathcal{N} = 1$ supergravity in the rigid limit in which the Planck mass is sent to infinity, setting the gravitini

to zero and requiring that supergravity variation is zero also, one finds the generalization of killing spinor equations, which involves the background auxiliary fields of the gravity multiplet.

$$\nabla\zeta = \mathcal{H}\zeta, \quad (5.14)$$

Where ζ is the supergravity parameter, now in the rigid limit, for \mathcal{H} we denote collectively the auxiliary fields of the gravity multiplet with the metric also fixed $g_{\mu\nu}$. This is reminiscent of the Witten topological twist [98], where an R -symmetry background field is added to cancel the torsion of the killing spinor. By imposing integrability conditions the classes of spaces admitting supersymmetry are restricted, the restriction comes basically from the relation of the gravity multiplet auxiliary fields and the curvatures of the space. The number of integrability conditions to impose will restrict further the space in resonance with the number of independent solutions one is asking the generalized killing spinor equation to have, namely the number of supersymmetries.

In this process one also gets the supersymmetrized Lagrangian on the curved manifold from the rigid limit of the supergravity Lagrangian. Still some care of this Lagrangians is needed, since one might end up with a non physical Lagrangian, like $\mathcal{N} = 1$ on S^4 (unless the theory is also conformal) [95, 99, 100]. This treatment has also been generalized partially to $\mathcal{N} = 2$ in 4D in [101, 102], and a classification of geometries admitting some of the $\mathcal{N} = 2$ supersymmetries found. There is a technical complication in the case of $\mathcal{N} = 2$ theories in comparison with $\mathcal{N} = 1$. In the later case one can get supersymmetric Lagrangians, and the supersymmetry algebra that closes off-shell on both vector and matter multiplets. In $\mathcal{N} = 2$ this not the case the supersymmetry algebra on the hypermultiplets comes usually on-shell and try to put the full set of supersymmetries off-shell might be just impossible. Luckily for the use of localization one just need a particular supercharge, and one can close this single supercharge off-shell like done in [70] to perform the computation. This technical problem can be also the reason why in $\mathcal{N} = 1$ case we have more general statements like the results for $\mathcal{N} = 1$ theories with R -symmetry in [82, 103].

6. 4D $\mathcal{N} = 2$ THEORIES ON $SU(2) \times U(1)$ ISOMETRIC SPHERE

This chapter is based on [104]. We follow the construction of Hama and Hosomichi [75], although initially implemented for the ellipsoid, the procedure can be implemented for more general backgrounds like [105] (See also [106, 107, 108] for slightly different approaches).

6.1 The Rigid Gravity limit: Killing Spinor Equation

The field content of the background $\mathcal{N} = 2$ gravity multiplet is given by the metric g_{mn} , a pair of tensors T_{mn} and \bar{T}_{mn} which are real, and self-dual or anti self-dual, an $SU(2)_R$ -adjoint vector V_{mAB} and an scalar M . Inspired by the rigid gravity limit the fermions of the gravity multiplet are set to zero and the gravitini variation provides the generalized Killing spinor equation [75],

$$D_l \Xi_A + \mathcal{T}_{mn} \gamma^{mn} \gamma_l \Xi_A = -i \gamma_l \Xi'_A, \quad (6.1)$$

Here $\Xi_A = \begin{pmatrix} \xi_{\alpha A} \\ \tilde{\xi}_{\dot{\alpha} A} \\ \xi_A \end{pmatrix}$, the covariant derivative D_m also contains the $SU(2)_R$, V_{mAB} background field

$$D_m \Xi_A = \nabla_m \Xi_A + i \Xi_B V_{mA}^B \quad \text{and} \quad \mathcal{T}_{mn} = \begin{pmatrix} T_{mn} & 0 \\ 0 & \bar{T}_{mn} \end{pmatrix},$$

Here the spinor Ξ'_A using (6.1) is

$$\Xi'_A = \frac{i}{4} \not{D} \Xi_A = \frac{i}{4} \Xi_{pA}, \quad (6.2)$$

In (6.4) we rise and lower spinor indices and R -Symmetry indices with $\epsilon^{\alpha\beta}$, $\epsilon^{\dot{\alpha}\dot{\beta}}$, ϵ^{AB} and their inverses, we have the conventions for them in (D) as well as the conventions for the covariant derivative ∇ and for the γ -matrices algebra .

Notice that in (6.1) we do not have $U(1)_R$ background field, even though $\mathcal{N} = 2$ theories usually have $SU(2)_R \times U(1)_R$ R -symmetry. As in [75] the reality condition (6.4) is not compatible with this $U(1)_R$, also when $\mathcal{T} \neq 0$, (6.1) breaks the $U(1)_R$ symmetry even if the reality condition is not enforced.

To guarantee the closure of the supersymmetry algebra in the general context, (6.1) is not enough and we need an auxiliary equation [75]:

$$\gamma^n D_n \Xi_{pA} + 4 D_l \mathcal{T}_{mn} \gamma^{mn} \gamma^l \Xi_A = M \Xi_A, \quad (6.3)$$

With M a background real scalar field, analogous to the one in [70]. We will be enforcing on Ξ_A the following reality conditions [75]

$$(\xi_{\alpha A})^\dagger = \xi^{A\alpha} = \epsilon^{\alpha\beta} \epsilon^{AB} \xi_{\beta B}, \quad (\tilde{\xi}^{\dot{\alpha}}_A)^\dagger = \tilde{\xi}^A_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon^{AB} \tilde{\xi}^{\dot{\beta}}_B, \quad (6.4)$$

These reality conditions will be important later for localization, they will guarantee the positive definiteness of bosonic part of \mathcal{Q} -exact regulator.

6.1.1 The Vector Multiplet

The field content of the vector multiplet is:

$$\begin{aligned} &\text{vector} \quad A_m, \quad \text{a pair of fermions} \quad \lambda_{\alpha A}, \tilde{\lambda}^{\dot{\alpha}}_A \\ &\text{two scalars} \quad \phi, \bar{\phi} \quad \text{and} \\ &\text{auxiliary scalar} \quad D_{AB} = D_{BA}, \end{aligned} \quad (6.5)$$

The vector and the scalar fields are not charged under the $SU(2)_R$ while the fermions are doublets, while the auxiliary field is a triplet. The supersymmetry transformation of a $U(1)$ vector multiplet is:

$$\begin{aligned} \mathcal{Q}A_m &= i\Xi^A \gamma_m \Lambda_A, \\ \mathcal{Q}\phi &= \Xi^A \lambda_A, \\ \mathcal{Q}\bar{\phi} &= \Xi^A \tilde{\lambda}_A, \\ \mathcal{Q}\Lambda_A &= -\frac{i}{2}(\gamma^{mn} \Xi_A F_{mn} + i8\gamma^{mn}(\tilde{\xi}_A \bar{T}_{mn} \phi + \xi_A T_{mn} \bar{\phi})) \\ &\quad + 2\gamma^m \xi_A D_m \bar{\phi} + 2\gamma^m \tilde{\xi}_A D_m \phi + \gamma^m (D_m \xi_A \bar{\phi} + D_m \tilde{\xi}_A \phi) + 2\Xi_A [\phi, \bar{\phi}] - \Xi^B D_{BA}, \\ \mathcal{Q}D_{AB} &= \Xi_A \gamma^m D_m \Lambda_B + \Xi_B \gamma^m D_m \Lambda_A + 2[\phi, \tilde{\xi}_A \tilde{\lambda}_B + \tilde{\xi}_B \tilde{\lambda}_A] - 2[\bar{\phi}, \xi_A \lambda_B + \xi_B \lambda_A]. \end{aligned} \quad (6.6)$$

We are using four component notation,

$$\begin{aligned} \Lambda_A &= \begin{pmatrix} \lambda_{\alpha A} \\ \tilde{\lambda}^{\dot{\alpha}}_A \end{pmatrix} \quad \text{abusing notation we write} \\ \lambda_A &= \begin{pmatrix} \lambda_{\alpha A} \\ 0 \end{pmatrix}, \quad \tilde{\lambda}_A = \begin{pmatrix} 0 \\ \tilde{\lambda}^{\dot{\alpha}}_A \end{pmatrix}, \quad \xi_A = \begin{pmatrix} \xi_{\alpha A} \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{\xi}_A = \begin{pmatrix} 0 \\ \tilde{\xi}^{\dot{\alpha}}_A \end{pmatrix} \end{aligned} \quad (6.7)$$

One can show that the supersymmetry closes by taking \mathcal{Q}^2 on each field, the answer for generic backgrounds, modulo signs and phases is the same as in section 2 of [75].

The supersymmetric YM action is given by the following expression:

$$S_{YM} = \frac{1}{g_{YM}^2} \int d^4x \sqrt{|g|} \mathcal{L}_{YM}, \quad (6.8)$$

where:

$$\begin{aligned} \mathcal{L}_{YM} = & \text{Tr} \left[\frac{1}{2} F^{mn} F_{mn} - i16 F^{mn} (\phi \bar{T}_{mn} + \bar{\phi} T_{mn}) - 64 \phi^2 \bar{T}^{mn} \bar{T}_{mn} \right. \\ & - 64 \bar{\phi}^2 T^{mn} T_{mn} + 4 D^m \bar{\phi} D_m \phi - 2 M \bar{\phi} \phi + 4 [\phi, \bar{\phi}]^2 + \frac{1}{2} D^{AB} D_{AB} \\ & \left. - \Lambda^A \gamma^m D_m \Lambda_A + 2 \lambda^A [\bar{\phi}, \lambda_A] - 2 \tilde{\lambda}^A [\phi, \tilde{\lambda}_A] \right], \end{aligned} \quad (6.9)$$

6.1.2 Hypermultiplet

r hypermultiplets contains the following scalar and fermion fields:

$$\begin{aligned} \text{scalars } q_{AI}, \quad \text{fermions } \Psi_I &= \begin{pmatrix} \psi_{\alpha I} \\ \bar{\psi}_{\dot{\alpha} I} \end{pmatrix}, \\ \text{and auxiliary fields } F_{AI} \quad \text{where } I &= 1, \dots, 2r. \end{aligned} \quad (6.10)$$

The I, J indices run from 1, ..., $2r$ these indices can be rise and lower with the Ω^{IJ} the real antisymmetric $sp(r)$ in variant metric. We have also the auxiliary scalar F_{IA} which transforms as a doublet under a local $SU(2)_{\tilde{R}}$ symmetry. This symmetry and the respective auxiliary field are introduced in the theory to impose the off-shell closure of the supersymmetry algebra on the matter multiplet, as introduce in [70], the algebra is off-shell just respect to the supersymmetry used to localize. Therefore the supersymmetry algebra on the matter multiplet will involve a further auxiliary spinor,

$$\hat{\Xi}_A = \begin{pmatrix} \hat{\xi}_{\alpha A} \\ \hat{\tilde{\xi}}_{\dot{\alpha} A} \end{pmatrix}$$

The off-shell supersymmetry transformation for the hypermultiplet is then:

$$\begin{aligned} \mathcal{Q} q_{AI} &= \Xi_A \Psi_I, \\ \mathcal{Q} \Psi_I &= 2 \gamma^m \Xi^A D_m q_{AI} + \gamma^m D_m \Xi^A q_{AI} - 4 \Xi_A \tilde{\Phi} q^A + 2 \hat{\Xi}^A F_{AI}, \\ \mathcal{Q} F_{AI} &= -\hat{\Xi}_A \gamma^m D_m \Psi_I - 2 \hat{\Xi}_A \Phi \Psi_I - 2 \hat{\Xi}_A \Lambda_B q_I^B - 2 \hat{\Xi}_A \mathcal{T}_{mn} \gamma^{mn} \Psi_I, \end{aligned} \quad (6.11)$$

Here $\Phi = \begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}$ and by $\hat{\Xi}_A \Phi \Psi_I$ we mean the product $\begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix} \hat{\Xi}_A \Psi_I$ with the gauge indices properly contracted, similarly $\Xi_A \tilde{\Phi} q^A$ means the product $(\sigma_1 \Phi \sigma_1) \Xi_A q^A$ with the gauge indices properly contracted and σ_1 the first Pauli matrix. The $SU(2)_{\tilde{R}}$ is not an independent symmetry, there is some alignment and the spinor $\hat{\Xi}_A$ has to satisfy the following algebraic relations as written in [75]:

$$\begin{aligned} \xi_A \hat{\xi}_B - \tilde{\xi}_A \hat{\xi}_B &= 0, \\ \xi^A \xi_A + \hat{\xi}^A \hat{\xi}_A &= 0, \\ \tilde{\xi}^A \tilde{\xi}_A + \hat{\xi}^A \hat{\xi}_A &= 0, \\ \xi^A \gamma^m \tilde{\xi}_A + \hat{\xi}^A \gamma^m \hat{\xi}_A &= 0, \end{aligned} \quad (6.12)$$

Then the action for the matter multiplet sector is given by:

$$S_{mat} = \int d^4x \sqrt{|g|} \mathcal{L}_{mat}, \quad (6.13)$$

where;

$$\begin{aligned} \mathcal{L}_{mat} = & \frac{1}{2} D_m q^A D^m q_A + q^A \{\phi, \bar{\phi}\} q_A + \frac{1}{2} q^A D_{AB} q^B + \frac{1}{8} (R + M) q^A q_A - \frac{i}{2} \bar{\psi} \bar{\sigma}^m D_m \psi \\ & - \frac{1}{2} \psi \phi \psi - \frac{1}{2} \bar{\psi} \bar{\phi} \bar{\psi} - \frac{1}{2} \psi \sigma^{kl} T_{kl} \psi + \frac{1}{2} \bar{\psi} \bar{\sigma}^{kl} \bar{T}_{kl} \bar{\psi} - q^A \lambda_A \psi - \bar{\psi} \bar{\lambda} q^A - \frac{1}{2} F^A F_A. \end{aligned} \quad (6.14)$$

6.2 Supersymmetry on the Squashed S^4

The $SU(2) \times U(1)$ -isometric S^4 (we will refer also to it as squashed S^4) which we will consider is defined by the following metric or vielbein one-forms:

$$\begin{aligned} ds^2 &= dr^2 + \frac{f(r)^2}{4} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{h(r)^2}{4} (d\psi + \cos \theta d\phi)^2, \\ e^4 &= dr, \quad e^3 = -\frac{h(r)}{2} (d\psi + \cos \theta d\phi), \quad e^2 = \frac{f(r)}{2} (\sin \psi d\theta - \sin \theta \cos \psi d\phi), \\ e^1 &= -\frac{f(r)}{2} (\cos \psi d\theta + \sin \theta \sin \psi d\phi), \end{aligned} \quad (6.15)$$

where $f(r)$ and $h(r)$ are smooth arbitrary functions of r . The above metric has $SU(2) \times U(1)$ isometry. The spin connection is given by the following non-zero components Ω_m^{ab} ,

$$\begin{aligned} \Omega_1^{21} &= 1 - \frac{h(r)^2}{2f(r)^2}, \quad \Omega_1^{43} = \frac{h'(r)}{2}, \quad \Omega_2^{31} = \frac{h(r) \sin(\psi)}{2f(r)}, \quad \Omega_2^{32} = \frac{h(r) \cos(\psi)}{2f(r)}, \\ \Omega_2^{41} &= \frac{1}{2} \cos(\psi) f'(r), \quad \Omega_2^{42} = -\frac{1}{2} \sin(\psi) f'(r), \quad \Omega_3^{21} = \cos(\theta) - \frac{h(r)^2 \cos(\theta)}{2f(r)^2}, \\ \Omega_3^{31} &= -\frac{h(r) \sin(\theta) \cos(\psi)}{2f(r)}, \quad \Omega_3^{32} = \frac{h(r) \sin(\theta) \sin(\psi)}{2f(r)}, \quad \Omega_3^{41} = \frac{1}{2} \sin(\theta) \sin(\psi) f'(r), \\ \Omega_3^{42} &= \frac{1}{2} \sin(\theta) \cos(\psi) f'(r), \quad \Omega_3^{43} = \frac{1}{2} \cos(\theta) h'(r), \end{aligned} \quad (6.16)$$

where $a, b = 1, \dots, 4$ are flat indices and $m = 1, \dots, 4$ is curved space index.

6.2.1 Solution of Killing Spinor Equation on the Squashed S^4

The purpose of this section is to show that if the background fields $(V_m)_B^A, T_{mn}, \bar{T}_{mn}, M$ are chosen appropriately, the $SU(2) \times U(1)$ isometric S^4 admits a Killing spinor which

is solution of the two sets of Killing spinor (6.1) and auxiliary (6.3) equations. We write the backgrounds T and V in a complexified version:

$$V_m = \begin{pmatrix} iv_{3m} & i(v_{1m} + iv_{2m}) \\ i(v_{1m} - iv_{2m}) & -iv_{3m} \end{pmatrix},$$

$$T = \begin{pmatrix} it_3 & i(t_1 - it_2) & 0 & 0 \\ i(t_1 + it_2) & -it_3 & 0 & 0 \\ 0 & 0 & it_3 & i(\bar{t}_1 - i\bar{t}_2) \\ 0 & 0 & i(\bar{t}_1 + i\bar{t}_2) & -i\bar{t}_3 \end{pmatrix}. \quad (6.17)$$

We will consider the following ansatz for the Killing spinor and we will calculate the background fields T , V and M such that this ansatz satisfies the set of Killing spinor equations

$$\xi = \begin{pmatrix} s_1(r) & 0 \\ 0 & t_2(r) \\ s_3(r) & 0 \\ 0 & t_4(r) \end{pmatrix}. \quad (6.18)$$

The Killing spinor satisfies the reality condition given in [75]:

$$(\xi_{\alpha A})^\dagger = \epsilon^{AB} \epsilon^{\alpha\beta} \xi_{\beta B}, \quad (\bar{\xi}_{\dot{\alpha} A})^\dagger = \epsilon^{AB} \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\xi}_{\dot{\beta} B}. \quad (6.19)$$

The parameters in the Killing spinor are arbitrary smooth functions of r . After solving the Killing spinor equations, it turns out that some of these parameters are constrained.

The general solution to the main and auxiliary equations using the ansatz (6.18) takes the following form:

$$s_1(r) = s(r), \quad s_3(r) = \frac{i c h(r)}{s(r)}, \quad t_2(r) = s(r),$$

$$t_4(r) = -\frac{i c h(r)}{s(r)},$$

$$t_3 = \frac{s(r) (f(r) (2f(r)s'(r) - s(r)f'(r)) + h(r)s(r))}{4cf(r)^2h(r)},$$

$$\bar{t}_3 = \frac{c (f(r)h(r) (s(r)f'(r) + 2f(r)s'(r)) - 2f(r)^2s(r)h'(r) + h(r)^2s(r))}{4f(r)^2s(r)^3},$$

$$v_{33} = \frac{1}{2} \left(\frac{h(r)}{f(r)^2} + \frac{h'(r) - 2}{h(r)} - \frac{2s'(r)}{s(r)} \right),$$

$$M = \frac{2f''(r)}{f(r)} + \frac{f'(r)^2 - 2h'(r) + \frac{4h(r)s'(r)}{s(r)}}{f(r)^2} + \frac{h(r)^2}{f(r)^4} + \frac{4s'(r) (s(r)h'(r) - h(r)s'(r))}{h(r)s(r)^2}. \quad (6.20)$$

Here only the non-zero part of the background fields and Killing spinor components are given, c is a real arbitrary constant which sets normalization of the killing vector we will to localize, $s(r)$ is a smooth function of r and the background fields T and

V_m are indexed by flat tangent space indices. For these background fields to be well defined on the squashed S^4 , it is necessary that $s(r)$ has no zero between the two poles. We thus determined the form of all the additional background fields in order for $\mathcal{N} = 2$ SUSY to be preserved on the squashed four-sphere. We have set $v_{12} = 0$, this choice of background preserves $SU(2) \times U(1) \times U(1)_R$ symmetry. Should we take $v_{12} \neq 0$ it can be shown that the symmetry is reduced to $SU(2) \times U(1)'$ where $U(1)' \equiv (U(1) \times U(1)_R)_{diagonal}$.

6.2.2 Regularity of the Background Fields

Our metric should look like the round S^4 at the North and South poles, this implies that $f(r) = h(r) = 0$ at $r = 0$ and $r = \pi$. Moreover for our metric to be non-singular in the interval $\pi > r > 0$, the functions $f(r)$ and $h(r)$ are strictly non-zero and do not change sign inside the interval.

North pole ($r = 0$): Near the North pole the regularity of invariant quantities R , $R_{\mu\nu}R^{\mu\nu}$ and of the background fields both in flat tangent space indices and curved space indices, fixes $f(r)$, $h(r)$ and $s(r)$ in the following form:

$$\begin{aligned} h(r) &= r + h_{n_3}r^3 + O(r^4), \\ f(r) &= r + f_{n_3}r^3 + O(r^4), \\ s(r) &= s_{n_0} + s_{n_2}r^2 + s_{n_3}r^3 + O(r^4). \end{aligned} \tag{6.21}$$

There are higher order terms, but those are irrelevant to the present analysis.

South pole ($r = \pi$): Similarly near the South pole the regularity requirements fix $f(r)$, $h(r)$ and $s(r)$ in the following way

$$\begin{aligned} h(r) &= \pi - r + h_{s_3}(\pi - r)^3 + O((\pi - r)^4), \\ f(r) &= \pi - r + f_{s_3}(\pi - r)^3 + O((\pi - r)^4), \\ s(r) &= (\pi - r)s_{s_1} + (\pi - r)^3s_{s_3} + O((\pi - r)^4). \end{aligned} \tag{6.22}$$

Where $h_{n_3}, f_{n_3}, s_{n_0}, s_{n_2}, s_{n_3}, h_{s_3}, f_{s_3}, s_{s_1}, s_{s_3}$ are arbitrary real constants.

For reasons that will become clear later, a quantity of interest which we want to calculate is $(s(r)^2 - \frac{c^2 h(r)^2}{s(r)^2})$. At the North pole it evaluates to $s_{n_0}^2$, whereas at the South pole it evaluates to $-\frac{c^2}{s_{s_1}^2}$. So it has the interesting property that it changes sign between North and South poles and hence passes through zero. This result will have important consequences later on, in section (6.4) when we will calculate the one-loop determinant, where we show that the relevant differential operators are transversally elliptic. Before proceeding, we want to comment that there is an ambiguity in the choice of the functions $f(r)$, $h(r)$ and $s(r)$ at the North and South poles, that is, if we take following choice for these functions at the North pole

$$\begin{aligned} h(r) &= -r + h_{n_3}r^3 + O(r^4), \\ f(r) &= r + f_{n_3}r^3 + O(r^4), \\ s(r) &= s_{n_1} + s_{n_3}r^3 + O(r^4), \end{aligned} \tag{6.23}$$

and the following choice at the South pole

$$\begin{aligned} h(r) &= r - \pi + h_{s_3}(\pi - r)^3 + O((\pi - r)^4), \\ f(r) &= \pi - r + f_{s_3}(\pi - r)^3 + O((\pi - r)^4), \\ s(r) &= s_{s_0} + s_{s_2}(\pi - r)^2 + s_{s_3}(\pi - r)^3 + O((\pi - r)^4), \end{aligned} \quad (6.24)$$

all the background fields are still regular there. The only difference is that the quantity $(s(r)^2 - \frac{c^2 h(r)^2}{s(r)^2})$ evaluates to $-\frac{c^2}{s_{n_1}^2}$ at the South pole and to $s_{s_0}^2$ at the North pole. Every other result remains the same.

6.2.3 Closure of the Supercharge Algebra

For localization computation we need to identify a continuous fermionic symmetry \mathcal{Q} and the corresponding Killing spinor is taken to be commuting. The supersymmetry transformation Q acting on the fields of $\mathcal{N} = 2$ SUSY theory squares into a combination of bosonic symmetries:

$$\begin{aligned} \mathcal{Q}^2 \equiv \mathcal{L}_v + Gauge(\hat{\Phi}) + Lorentz(L_{ab}) + Scale(\omega) \\ + R_{U(1)}(\Theta) + R_{SU(2)}(\hat{\Theta}_{AB}) + \check{R}_{SU(2)}(\hat{\check{\Theta}}), \end{aligned} \quad (6.25)$$

with various parameters defined as in [75]. For the vector multiplet the SUSY algebra is closed off shell, the only requirement being that the Killing spinor equations be satisfied. For the hypermultiplet the closure of full $\mathcal{N} = 2$ SUSY algebra requires the existence of infinite number of auxiliary spinors and auxiliary fields. But for localization computation we need only one supercharge corresponding to a particular Killing spinor and in this case only finite number of auxiliary spinors are required. These auxiliary spinors are required to satisfy certain constraint equations (see [70]). Next we compute these transformation parameters for our background. First of all, we observe that $\xi^A \xi_{pA} = \bar{\xi}^A \bar{\xi}_{pA} = 0$. This condition implies that $\omega = \Theta = 0$. In other words the square of the supersymmetry transformation does not give rise to dilation or $U(1)_R$ transformation. This condition is necessary because the non-zero values of the background fields T_{ab} and \bar{T}_{ab} break the $U(1)_R$ symmetry anyway.

The explicit expression for other transformation parameters are given below

$$\begin{aligned}
L_{ab} &= \begin{pmatrix} 0 & -8c & 0 & 0 \\ 8c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Theta_{AB} &= \begin{pmatrix} 0 & 2c \left(\frac{h(r)^2}{f(r)^2} - \frac{2s'(r)h(r)}{s(r)} + h'(r) \right) \\ 2c \left(\frac{h(r)^2}{f(r)^2} - \frac{2s'(r)h(r)}{s(r)} + h'(r) \right) & 0 \end{pmatrix}, \\
\hat{\Theta}_B^A &= \begin{pmatrix} 4c & 0 \\ 0 & -4c \end{pmatrix}, \\
\mathcal{L}_v \xi &= \begin{pmatrix} -\frac{2cs(r)((h'(r)-2)f(r)^2+h(r)^2)}{f(r)^2} & 0 \\ 0 & \frac{2cs(r)((h'(r)-2)f(r)^2+h(r)^2)}{f(r)^2} \\ \frac{2ic^2h(r)(f(r)^2(h'(r)+2)-h(r)^2)}{f(r)^2s(r)} & 0 \\ 0 & \frac{2ic^2h(r)(f(r)^2(h'(r)+2)-h(r)^2)}{f(r)^2s(r)} \end{pmatrix},
\end{aligned} \tag{6.26}$$

where the Lie derivative Lie_v is defined as $L_v \xi \equiv v^m D_m \xi + \frac{1}{4} D_{[a} v_{b]} \Gamma^{ab} \xi$. The non-zero L_{ab} implies the fact that the $U(1)$ group which is used to find the fixed points of the manifold, belongs to the Cartan of $SU(2)$ part of the isometry group $SU(2) \times U(1)$. Therefore it follows that our Killing spinor is invariant under Q^2 . In 4-component notation:

$$Q^2 \xi = i \mathcal{L}_v \xi - \xi \hat{\Theta} = 0. \tag{6.27}$$

The auxiliary spinor, which helps to close off-shell the supersymmetry, is given by:

$$\check{\xi} = \begin{pmatrix} \frac{ch(r)}{s(r)} & 0 \\ 0 & \frac{ch(r)}{s(r)} \\ -is(r) & 0 \\ 0 & is(r) \end{pmatrix}. \tag{6.28}$$

To fix the background $SU(2)_{\tilde{R}}$, we have to fix the corresponding gauge field \check{V}_m :

$$\check{V}_m = \begin{pmatrix} i\check{v}_{3m} & i(\check{v}_{1m} + i\check{v}_{2m}) \\ i(\check{v}_{1m} - i\check{v}_{2m}) & -i\check{v}_{3m} \end{pmatrix}. \tag{6.29}$$

The requirement that all the background fields be invariant under the action of Q^2 fixes all the components of \check{V}_m to zero except $\check{v}_{33}, \check{v}_{34}$, which remain arbitrary.

After the gauge fixing, $\hat{\Theta}_B^A$ becomes

$$\hat{\Theta}_B^A = \begin{pmatrix} -4(h(r)\check{v}_{33}(r)c + c) & 0 \\ 0 & 4(h(r)\check{v}_{33}(r)c + c) \end{pmatrix}. \tag{6.30}$$

And also the auxiliary spinor $\check{\xi}$ is proven to be invariant under Q^2

$$Q^2 \check{\xi} = 0. \tag{6.31}$$

6.3 Localization

6.3.1 Contour of Integration

Now we want to compute explicitly the path integral, in euclidean signature all the fields are complexified, but in the Lagrangian and measure we have just the fields and not its complex conjugates, so we have actually to specify a contour of integration.

Vector Multiplet

$$(A_\mu)^\dagger = A_\mu, (\phi)^\dagger = \bar{\phi}, (\lambda_{\alpha A})^\dagger = \lambda^{\alpha A}, (\tilde{\lambda}_{\dot{\alpha} A})^\dagger = \tilde{\lambda}^{\dot{\alpha} A}, (D_{AB})^\dagger = \epsilon^{AC} \epsilon^{BD} D_{CD}, \quad (6.32)$$

Matter Multiplet

$$(q_{AI})^\dagger = \Omega^{IJ} \epsilon^{AB} q_{BJ}, (\psi_{\alpha I})^\dagger = \Omega^{IJ} \epsilon^{\alpha\beta} \psi_{\beta J}, (\bar{\psi}_{\dot{\alpha} I})^\dagger = \Omega^{IJ} \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\beta} J}, \\ (F_{AI})^\dagger = \Omega^{IJ} \epsilon^{AB} F_{BJ}. \quad (6.33)$$

This 'reality conditions' pick a particular contour of integration such that the \mathcal{Q} -deformation we will add to localize is semi-positive definite.

6.3.2 S_{YM} Saddle Points

The path integral computation of the expectation value of an observable of a supersymmetric YM theory which is invariant under a supercharge \mathcal{Q} localizes to a subset $S_{\mathcal{Q}}$ of the entire field space. The zero locus of the supercharge \mathcal{Q} coincides with the set of bosonic configurations for which the supersymmetry variations of the fermions vanish:

$$\mathcal{Q}\Psi = 0 \quad \text{for all fermions } \Psi. \quad (6.34)$$

This is easily seen if we can take as regulator the \mathcal{Q} -exact deformation: $\mathcal{Q}\mathcal{V} = \mathcal{Q}((\mathcal{Q}\Psi)^\dagger \Psi)$.

To take into account the gauge fixing, the supercharge \mathcal{Q} is generalized to $\hat{\mathcal{Q}} \equiv \mathcal{Q} + \mathcal{Q}_B$, where \mathcal{Q}_B is the BRST-supercharge. However as pointed out in [70], this does not affect the zero locus. To effectively calculate the zero locus of the supercharge, we add to the Lagrangian a \mathcal{Q} -exact quantity $\mathcal{Q}\mathcal{V}$, whose critical point set is $S_{\hat{\mathcal{Q}}}$ and whose bosonic part is semi-positive definite. Now either solving the localization equation

$$\hat{\mathcal{Q}}\lambda = 0, \quad (6.35)$$

directly or analyzing the $\hat{\mathcal{Q}}$ -transform of the following quantity,

$$\mathcal{V} = \text{Tr}[(\hat{\mathcal{Q}}\lambda_{\alpha A})^\dagger \lambda_{\alpha A} + (\hat{\mathcal{Q}}\bar{\lambda}_{\dot{\alpha} A})^\dagger \bar{\lambda}_{\dot{\alpha} A}], \quad (6.36)$$

which has semi-positive definite bosonic part. In writing explicitly (6.36) we use the proper reality conditions which make the action well defined. We get the analogous expression to the equation (4.2) in [75].

Analyzing that expression we get the following partial differential equations for $\phi - \bar{\phi} \equiv \phi_2(\psi, \theta, \varphi, r)$, where we make use of Bianchi identities to get the second equation:

$$\partial_\psi \phi_2(\psi, \theta, \varphi, r) = 0, \quad (6.37)$$

and

$$\tilde{\nabla}^2 \phi_2(\theta, \varphi, r) + \frac{f(r)^2}{2h(r)} \xi \Gamma^m \xi_p \partial_m \phi_2(\theta, \varphi, r) + G(r) \phi_2(\theta, \varphi, r) = 0, \quad (6.38)$$

where in the second equation we used the fact that $\phi_2(\psi, \theta, \varphi, r)$ is independent of ψ -coordinate. $\tilde{\nabla}^2$ is the following Laplacian like operator:

$$\tilde{\nabla}^2_* = \frac{f(r)^2}{2h(r)} \frac{h(r)}{\sqrt{g} f(r) \xi_n} \nabla_\mu \left(\sqrt{g} \xi_n^2 g^{\mu\nu} \nabla_\nu \left(\frac{f(r)}{h(r)} * \right) \right) \quad (6.39)$$

$\xi_n = \xi \cdot \xi$ is the proper norm of the four component spinor and $G(r)$

$$\begin{aligned} G(r) = & \frac{1}{h(r)^3 s(r)^3} \left(-c^2 h(r)^4 (s(r) (f'(r)^2 + 2h'(r)) - 2f(r)f'(r)s'(r)) \right. \\ & - h(r)^2 (-3c^2 f(r)^2 s(r)h'(r)^2 + 2f(r)s(r)^4 f'(r)s'(r) + s(r)^5 f'(r)^2) \\ & + h(r)^3 (c^2 f(r)^2 s(r)h''(r) + 2s'(r) (s(r)^4 - 2c^2 f(r)^2 h'(r))) + 2c^2 h(r)^5 s'(r) \\ & + f(r)h(r)s(r)^4 (2h'(r) (s(r)f'(r) + 2f(r)s'(r)) + f(r)s(r)h''(r)) \\ & \left. - f(r)^2 s(r)^5 h'(r)^2 \right) \end{aligned} \quad (6.40)$$

For the round sphere

$$f(r) = \sin r, \quad h(r) = \sin r, \quad s(r) = \frac{1}{\sqrt{2}} \cos\left(\frac{r}{2}\right), \quad (6.41)$$

the field $\phi_2 = 0$ at the localization locus, which will also ensure that $A_m = 0$ at the locus. This result is true in an open neighborhood of the round S^4 , as appears also in [75], and so we will assume it is the solution to the locus equations.

The saddle points are thus labeled by a Lie Algebra valued constant a_0 , and are given by the equations[70, 75]:

$$A_m = 0, \quad \phi = \bar{\phi} = a_0, \quad D_{AB} = -ia_0 \omega_{AB}, \quad (6.42)$$

The value of the Super-Yang-Mills action on this saddle point is then:

$$\frac{1}{g_{YM}^2} \int d^4x \sqrt{g} L_{YM}|_{\text{saddlepoint}} = \frac{2\pi^3 \text{Tr}[a_0^2]}{c^2 g_{YM}^2}. \quad (6.43)$$

6.3.3 Saddle points for Matter multiplet

To find the saddle points of the matter multiplet we will use the following fermionic functional

$$\mathcal{V}_{mat} = \text{Tr}[(\hat{Q}\psi_{\alpha I})^\dagger \psi_{\alpha I} + (\hat{Q}\bar{\psi}_I^{\dot{\alpha}})^\dagger \bar{\psi}_I^{\dot{\alpha}}]. \quad (6.44)$$

The bosonic part of $\hat{Q}\mathcal{V}_{mat}$ is

$$\hat{Q}\mathcal{V}_{mat}|_{bos} = \text{Tr}[(\hat{Q}\psi_{\alpha I})^\dagger \hat{Q}\psi_{\alpha I} + (\hat{Q}\bar{\psi}_I^{\dot{\alpha}})^\dagger \hat{Q}\bar{\psi}_I^{\dot{\alpha}}]. \quad (6.45)$$

It is easy to check that:

$$\hat{Q}\mathcal{V}_{mat}|_{bos} = 4\|\xi\|^2 \left(\frac{1}{2}(D_m q^{AI} - P_m q^{AI})^2 + M_q(r) q^{AI} q_{IA} - \frac{1}{2} F^{AI} F_{IA} \right), \quad (6.46)$$

where

$$P_{mA}^B = \frac{1}{\|\xi\|^2} (2(\epsilon \xi \gamma_m \xi_p + \epsilon \xi T \gamma_m \xi)^B{}_A + D^n \text{Log}(\|\xi\|^2) (\epsilon \xi \gamma_{nm} \xi)^B{}_A), \quad (6.47)$$

and

$$\begin{aligned} M_q = & -\frac{1}{4}R + \frac{1}{\|\xi\|^2} (8\xi_p^A \xi_{pA} + \xi^A \gamma^m T^2 \gamma_m \xi_A - D^n \text{Log}(\|\xi\|^2) \xi^A (3\gamma_m \xi_{pA} + T \gamma_m \xi_A) + \\ & \frac{1}{2}(P^{mA}{}_B P_{mA}^B)) - \frac{1}{2\|\xi\|^2} P^{mA}{}_A P_{mB}^B, \end{aligned} \quad (6.48)$$

where $\xi_A = (\xi_{\alpha A}, \bar{\xi}_{\dot{\alpha} A}), \epsilon^{AB}$ is the $SU(2)_R$ tensor and R is the Ricci scalar. As a result of the condition $F_{IA}^\dagger = -F^{AI}$ which is imposed along the contour of path integration, all the bosonic terms are manifestly positive definite, except the term containing $M_q(r)$. For the round S^4

$$M_q(r) = \frac{7}{8} + \frac{\cos(2r)}{8}, \quad (6.49)$$

and it is bounded from below by $\frac{3}{4}$. Therefore there is a large open neighborhood of the round sphere for which $M_q(r)$ is positive definite. So we get the result for the saddle points of the hypermultiplet as

$$q_{IA} = 0, \quad F_{IA} = 0. \quad (6.50)$$

Hence there will be no classical contribution from the hypermultiplet sector.

6.4 One-loop determinant

To calculate the one-loop determinant we have to first fix the gauge. We choose the following gauge function[75].

$$G = i\partial_m A^m + iL_v((\xi^A \xi_A - \bar{\xi}_A \bar{\xi}^A)\phi_2 - v^m A_m). \quad (6.51)$$

The saddle point conditions do not change under the new supercharge $\hat{Q}^2 \equiv (Q_B + Q)^2$, with the zero mode of $\phi_1 = a_0$ at the saddle point.

6.4.1 Vector Multiplet Contribution

The basic idea of localization is that the actual value of the path integral or any other \hat{Q} -closed observable remains unchanged under any \hat{Q} -exact deformation $\mathcal{L} \rightarrow \mathcal{L} + t\hat{Q}(\mathcal{V} + \mathcal{V}_{GF})$. By choosing the bosonic part of $\mathcal{L} \rightarrow \mathcal{L} + t\hat{Q}(\mathcal{V} + \mathcal{V}_{GF})$ positive definite and sending $t \rightarrow \infty$, Gaussian approximation becomes exact for the path integral over the fluctuations around the locus. The Gaussian integral evaluates to the square root of the ratio between the determinant of a fermionic kinetic operator $K_{Fermions}$ and that of a bosonic kinetic operator K_{Bosons} . These kinetic operators coming from the quadratic part of the \hat{Q} -exact regulator.

To compute the 1-loop contribution it is convenient to change variables in the path integral to a set, X, Ξ , which makes manifest the cohomology of \hat{Q} [70, 75]. After doing that, the quadratic part of $\mathcal{V} + \mathcal{V}_{GF}$ can be written as:

$$(\mathcal{V} + \mathcal{V}_{GF})|_{quadratic} = (\hat{Q}\mathbf{X}, \Xi) \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \hat{Q}\Xi \end{pmatrix}, \quad (6.52)$$

where D_{ij} are differential operators and X, Ξ are cohomologically paired bosonic and fermionic fields respectively,

$$\Xi \equiv (\Xi_{AB}, \bar{C}, C), \quad \mathbf{X} = (\phi_2, A_m; \bar{a}_0, B_0), \quad (6.53)$$

and

$$\Xi_{AB} \equiv 2\bar{\xi}_{(A}\bar{\lambda}_{B)} - 2\xi_{(A}\lambda_{B)}, \quad (6.54)$$

where $\bar{C}, C, \bar{a}_0, B_0$ belong to the ghost multiplets. The fields X and Ξ can be regarded as sections of bundles E_0, E_1 over the squashed sphere and hence D_{10} acts on the complex as $D_{10} : \Gamma(E_0) \rightarrow \Gamma(E_1)$. The invariance of the deformation term $\hat{Q}(\mathcal{V} + \mathcal{V}_{GF})$ under the action of \hat{Q} and the pairing of the fields under $\hat{Q}^2 = \mathbf{H}$ leads to the cancellations between bosonic and fermionic fluctuations, which gives the following ratio [70, 75]:

$$\frac{\det_{Coker D_{10}} \mathbf{H}}{\det_{Ker D_{10}} \mathbf{H}}. \quad (6.55)$$

The fact that \hat{Q}^2 commutes with the differential operators D_{ij} is used in the derivation of the last expression and is a result of the invariance of $(\mathcal{V} + \mathcal{V}_{GF})$ under \hat{Q}^2 .

6.4.2 Index of D_{10}

To evaluate the ratio (6.55) through the index computation, we first note that the constant fields B_0, \bar{a}_0 have each weight 0 under the action of $U(1)$ at the poles and are thus regarded as sitting in the kernel of D_{10} and making a contribution of 2. For the contribution of other fields we need an explicit expression for D_{10} ¹, which

¹ Strictly speaking the relevant differential operator for the index computation is a combination of the original D_{10} and D_{11} . But it turns out that this operator commutes with \mathbf{H} and the distinction becomes irrelevant.

is read from equation (E.1) To compute the index of D_{10} it is better to use its, symbol $\sigma(D_{10})$, this is computed by taking the Fourier transform of the operator D_{10} and then retaining only the highest order derivative (momentum) terms [70]. To write the symbol explicitly we have to express the Fourier transform of D_{10} in the following orthonormal basis of four unit vector fields μ_a^m ($a = 1, 2, 3, 4$), which relabels the original vielbein basis

$$-2i(\tau^a)_B^A \bar{\xi}^B \bar{\sigma}^m \xi_A = 4ch(r)\mu_a^m, \quad 2\bar{\xi}^A \bar{\sigma}^m \xi_A = 4ch(r)\mu_4^m, \quad (a = 1, 2, 3), \quad (6.56)$$

Here c is the constant appearing in the definition of the Killing spinor. So the symbol is given by:

$$\sigma(D_{10}) = \begin{pmatrix} p_4 W(r) & p_3 & -p_2 & -p_1 W(r) & -4cp_1 h(r) \\ -p_3 & p_4 W(r) & p_1 & -p_2 W(r) & -4cp_2 h(r) \\ p_2 & -p_1 & p_4 W(r) & -p_3 W(r) & -4cp_3 h(r) \\ p_1 & p_2 & p_3 & p_4 W(r)^2 & 4cp_4 h(r)W(r) \\ p_1 p_4 & p_2 p_4 & p_3 p_4 & p_4^2 - 8c(\sum_i^3 p_i^2)h(r) & 2(\sum_i^3 p_i^2)W(r) \end{pmatrix}, \quad (6.57)$$

where $W(r) \equiv 2s(r)^2 - \frac{2c^2 h(r)^2}{s(r)^2}$. This matrix can be block diagonalized in terms of 1×1 and 4×4 factors, the relevant part of the symbol to compute the index is the following 4×4 block ,

$$\sigma(D'_{10}) = \begin{pmatrix} p_4 W(r) & p_3 & -p_2 & -p_1 \\ -p_3 & p_4 W(r) & p_1 & -p_2 \\ p_2 & -p_1 & p_4 W(r) & -p_3 \\ p_1 & p_2 & p_3 & p_4 W(r) \end{pmatrix}. \quad (6.58)$$

The determinant of this symbol is:

$$\text{Det}(\sigma(D'_{10})) = \left(\frac{4c^4 p_4^2 h(r)^4}{s(r)^4} - 8c^2 p_4^2 h(r)^2 + p_1^2 + p_2^2 + p_3^2 + 4p_4^2 s(r)^4 \right)^2. \quad (6.59)$$

For $p_1 = p_2 = p_3 = 0$ and $p_4 \neq 0$, this value of determinant changes sign between North and South poles as discussed in section (6.2.2), hence it has at least one zero. Therefore the symbol is not invertible at the location of that zero and by definition D_{10} cannot be elliptic. But restricting the momentum to $p_4 = 0$, σ is always invertible provided (p_1, p_2, p_3) are not all zero simultaneously. Therefore D_{10} is a transversally elliptic operator with respect to the symmetry generated by v . In general the kernel and cokernel of such transversally elliptic operator are infinite dimensional, but since $[\hat{Q}^2, D_{ij}] = 0$, they can both be splitted into irreps. of \mathbf{H} with finite multiplicities, these multiplicities can be read off from the index theorem as explained in [70]. The index theorem localizes the contributions to the fixed points of the action of \mathbf{H} , that is to the North and South poles of the squashed S^4 . According to the Atiyah-Bott [?] formula, the index is given by,

$$\text{ind}(D'_{10}) = \sum_{x=\text{fixed points}} \frac{\text{Tr}_{E_0}(\gamma) - \text{Tr}_{E_1}(\gamma)}{\det(1 - \frac{\partial \tilde{x}}{\partial x})}, \quad (6.60)$$

where γ denotes the eigenvalue of the action of the operator $e^{i\mathbf{H}t}$ on the vector and $SU(2)_R$ indices of the fields. So we need the action of $e^{i\mathbf{H}t}$ Near the North and South poles, on the local coordinates $z_1 \equiv x_1 + ix_2, z_2 \equiv x_3 + ix_4$, where we are defining near the North pole:

$$\begin{aligned} x_1 + ix_2 &= r \cos\left(\frac{\theta}{2}\right) e^{i\frac{\psi+\varphi}{2}}, \\ x_3 + ix_4 &= r \sin\left(\frac{\theta}{2}\right) e^{i\frac{\psi-\varphi}{2}}, \end{aligned} \quad (6.61)$$

so,

$$z_1 \rightarrow e^{4ict} z_1 \equiv q_1 z_1, \quad z_2 \rightarrow e^{4ict} z_2 \equiv q_2 z_2, \quad (6.62)$$

With $0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi, 0 \leq \psi \leq 4\pi$ As for the action of Q^2 on the fields of vector multiplet, its eigenvalues turn out to be of the same form as in [75], except that in our case $q_1 = q_2 = q = e^{4ict}$. Putting all together, also the similar contribution from the South pole, we get the index D_{10} .

The one loop determinant can be computed by extracting the spectrum of eigenvalues of \mathbf{H} from the index. For a non-abelian group G , with a_0 in its Cartan sub algebra, the one loop contribution of the vector multiplet can be written as [75]:

$$\begin{aligned} Z_{1-loop}^{vec} &= \left(\frac{\det K_{fermion}}{\det K_{boson}} \right)^{\frac{1}{2}} = \prod_{\alpha \in \Delta_+} \frac{1}{(\hat{a}_0 \cdot \alpha)^2} \times \\ &\prod_{m,n \geq 0} ((m+n) + i\hat{a}_0 \cdot \alpha)((m+n+2) + i\hat{a}_0 \cdot \alpha)((m+n) - i\hat{a}_0 \cdot \alpha)((m+n+2) - i\hat{a}_0 \cdot \alpha) \\ &= \prod_{\alpha \in \Delta_+} \frac{\Upsilon_1(i\hat{a}_0 \cdot \alpha) \Upsilon_1(-i\hat{a}_0 \cdot \alpha)}{(\hat{a}_0 \cdot \alpha)^2}, \end{aligned}$$

where $\hat{a}_0 \equiv \frac{a_0}{4c}$. The function $\Upsilon(x)$ has zeros at $x = -(m+n), (m+n+2)$, this function is implemented to regularized the infinite products. It is defined by:

$$\Upsilon_b(x) = \prod_{n_1, n_2 \geq 0} (bn_1 + \frac{n_2}{b} + x)(bn_1 + \frac{n_2}{b} + b + \frac{1}{b} - x), \quad (6.63)$$

where b is a constant that in the case of [75] is exactly the squashing parameter, while and in our case $b = 1$.

6.4.3 Hypermultiplet one-loop contribution

We begin also with cohomological pairing [70, 75] for the matter sector, the computation of the one-loop determinant reduces to that of the index of an operator D_{10}^{mat} . This operator corresponds to the terms bilinear in the fields Ξ and q_{IA} in the functional \mathcal{V}_{mat} . Its symbol $\sigma(D_{10}^{mat})$ is given by

$$\sigma(D_{10}^{mat}) = \begin{pmatrix} \frac{2((p_3 - ip_4)s(r)^4 + c^2 h(r)^2(p_3 + ip_4))}{s(r)^4 + c^2 h(r)^2} & 2(p_1 + ip_2) \\ 2((p_1 - ip_2)) & -\frac{2((p_3 + ip_4)s(r)^4 + c^2 h(r)^2(p_3 - ip_4))}{s(r)^4 + c^2 h(r)^2} \end{pmatrix}. \quad (6.64)$$

The determinant of this symbol is

$$\text{Det}[\sigma(D_{10}^{mat})] = -\frac{4(s(r)^4 - c^2 h(r)^2)^2}{(c^2 h(r)^2 + s(r)^4)^2} p_4^2 - 4p_1^2 - 4p_2^2 - 4p_3^2. \quad (6.65)$$

For $p_1 = p_2 = p_3 = 0, p_4 \neq 0$, the determinant changes sign somewhere between North and South poles (see section (6.2.2)) and hence it possesses at least one zero. Therefore the operator D_{10}^{mat} is again transversally elliptic with respect to the isometry generated by L_v in the p_4 direction.

The index for the action of \mathbf{H} on different fields at the poles can be calculated by using Atiyah-Bott formula. With $q_1 = q_2 = e^{4ict}$ in our case of squashed S^4 , the eigenvalues for the action of Q^2 on the matter multiplet case again turn out to have the same form as in [75].

For the hypermultiplets coupled to gauge symmetry, in the representation $R \oplus \bar{R}$ the final result for the one-loop determinant for the hypermultiplets becomes:

$$\begin{aligned} Z_{1-loop}^{hyp} &= \prod_{\rho \in R} \prod_{m, n \geq 0} ((m + n + 1) - i\hat{a}_0 \cdot \alpha)^{-1} ((m + n + 1) + i\hat{a}_0 \cdot \alpha)^{-1} \\ &= \prod_{\rho \in R} \Upsilon_1(i\hat{a}_0 \cdot \rho + 1)^{-1}. \end{aligned} \quad (6.66)$$

where ρ runs over all the weights in a given representation.

6.5 Instanton contribution

Near the North pole the Killing spinor evaluates to

$$\xi = \begin{pmatrix} s_{n_0} & 0 \\ 0 & s_{n_0} \\ \frac{icr}{s_{n_0}} & 0 \\ 0 & -\frac{icr}{s_{n_0}} \end{pmatrix}, \quad (6.67)$$

so that $\xi^A \xi_A = 2s_{n_0}^2$ and $\bar{\xi}_A \bar{\xi}^A = \frac{2c^2 r^2}{s_{n_0}^2}$. Since $\bar{\xi}_A \bar{\xi}^A \rightarrow 0$ at the North pole, the localization equation has to be evaluated away from the North pole to have smooth gauge field configurations.

Similarly near the South pole

$$\xi = \begin{pmatrix} (\pi - r)s_{s_1} & 0 \\ 0 & (\pi - r)s_{s_1} \\ \frac{ic}{s_{s_1}} & 0 \\ 0 & -\frac{ic}{s_{s_1}} \end{pmatrix}, \quad (6.68)$$

and $\xi^A \xi_A = 2(\pi - r)^2 s_{s_1}^2$ and $\bar{\xi}_A \bar{\xi}^A = \frac{2c^2}{s_{s_1}^2}$. In this case $\xi^A \xi_A \rightarrow 0$. Therefore South pole has also to be excluded if smooth gauge field configurations are assumed.

To include the contribution from the poles, we first notice that because $\bar{\xi}_A \bar{\xi}^A \rightarrow 0$ at the North pole, in general $F_{mn}^+ \neq 0, F_{mn}^- = 0$ there and still solve the localization equation. These configurations are the pointlike anti-instantons contribution.

Also at the North pole the following condition is satisfied for our background

$$\frac{1}{4}\Omega_m^{ab}\sigma_{ab}\xi_A + i\xi_B V_{mA}^B = 0. \quad (6.69)$$

Likewise, at the South pole $\xi^A \xi_A \rightarrow 0$, and we get the point instanton contribution $F_{mn}^+ = 0, F_{mn}^- \neq 0$ and the following twisting condition is satisfied

$$\frac{1}{4}\Omega_m^{ab}\bar{\sigma}_{ab}\bar{\xi}_A + i\bar{\xi}_B V_{mA}^B = 0. \quad (6.70)$$

The Killing vector near the North pole can be written as

$$v^m \frac{\partial}{\partial x_m} = 4c(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}) + 4c(x_3 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_3}). \quad (6.71)$$

Notice that near the South pole our $\mathcal{N} = 2$ theory on squashed S^4 approaches topologically twisted theory with Omega deformation parameters $\epsilon_1 = 4c, \epsilon_2 = 4c$ [42, 109], and the contribution of these point-instantons is given by $Z_{inst}(a_0, \epsilon_1, \epsilon_2, \tau)$, where the parameter τ is defined by $\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$.

Whereas near the North pole, the contribution of point anti-instantons is given by $Z_{inst}(a_0, \epsilon_1, \epsilon_2, \bar{\tau})$. Putting all together, the final form of the squashed S^4 partition function is

$$Z = \int d\hat{a}_0 e^{-\frac{2\pi^3 \text{Tr}[a_0^2]}{c^2 g_{YM}^2}} |Z_{inst}|^2 \prod_{\alpha \in \Delta_+} \Upsilon_1(i\hat{a}_0 \cdot \alpha) \Upsilon_1(-i\hat{a}_0 \cdot \alpha) \prod_{\rho \in R} \Upsilon_1(i\hat{a}_0 \cdot \rho + 1)^{-1}. \quad (6.72)$$

6.6 Final Remarks

We have computed the partition function of $\mathcal{N} = 2$ SUSY on squashed S^4 which admits $SU(2) \times U(1)$ isometry, using SUSY Localization technique. We find that the full partition function is independent of the squashing parameters as well as the other supergravity background fields.

The squashing functions independence of the one-loop part of the partition function, which is obvious from the form of the relevant Killing vector v , can perhaps be attributed to the fact that in our squashed S^4 the theory is topologically twisted at the poles. This is because the $SU(2)_R$ symmetry which is generically broken down to $U(1)_R$ on the squashed S^4 excluding the poles, is again enhanced to $SU(2)_R$ at the poles. So this $SU(2)_R$ can be identified at the poles with the $SU(2)$ Lorentz isometry to topologically twist the theory. The classical part can be written as a total derivative and gives to a contribution which is independent of the squashing parameters.

It will be interesting to explain this independence along the same lines given in [82]. That is to say, if we deform the vector multiplet and hypermultiplet actions around the round S^4 with respect to e.g. $f(r)$, it might be possible to write these deformed actions as \mathcal{Q} -exact terms separately. This \mathcal{Q} -exactness of the deformed action will explain the independence of partition function of the parameter $f(r)$ in the sense of [82]. However we have to consider perturbations around the round S^4 , unlike [82], where it is perturbed around flat R^4 .

APPENDIX

A. ALGEBRA CONVENTIONS FOR 3D CHERN-SIMONS THEORY

A.1 $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{sl}(3, \mathbb{R})$

The $\mathfrak{sl}(2, \mathbb{R})$ in the defining representation:

$$L_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad L_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad (\text{A.1})$$

satisfying the standard algebra:

$$[L_i, L_j] = (i - j)L_{i+j}, \quad (\text{A.2})$$

the killing metric (organizing the generators as L_{-1}, L_0, L_1) is given by:

$$g_{ab} = \text{tr}(t_a t_b) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ -1 & 0 & 0 \end{pmatrix}. \quad (\text{A.3})$$

The $\mathfrak{sl}(3)$ algebra used in (2.26)

$$\begin{aligned} L_{-1} &= \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ W_{-2} &= \begin{pmatrix} 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad W_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad W_0 = \frac{2}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ W_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (\text{A.4})$$

Satisfying the algebra:

$$\begin{aligned} [L_i, L_j] &= (i - j)L_{i+j}, \\ [L_i, W_m] &= (2i - m)W_{i+m}, \\ [W_n, W_m] &= -\frac{1}{3}(n - m)(2n^2 + 2m^2 - mn - 8)W_{n+m}, \end{aligned} \quad (\text{A.5})$$

The killing metric:

$$g_{ab} = \text{tr}(t_a t_b) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.6})$$

A.2 $\mathfrak{hs}(\lambda)$

The construction of the $\mathfrak{hs}(\lambda)$ algebra can be seen for example in [110]. The algebra is spanned by the set of generators V_t^s with $s = 2, \dots, \infty$ and $1 - s \leq t \leq s - 1$. The element V_0^1 denotes the identity operator. To define the algebra we use the \star -product representation constructed in [58]:

$$V_m^s \star V_n^t = \frac{1}{2} \sum_{i=1,2,3,\dots}^{s+t-\text{Max}[|m+n|, |s-t|]-1} g_i^{st}(m, n; \lambda) V_{m+n}^{s+n-i} \quad (\text{A.7})$$

With the constants:

$$g_i^{st}(m, n; \lambda) \equiv \frac{q^{i-2}}{2(i-1)!} {}_4F_3 \left[\begin{matrix} \frac{1}{2} + \lambda & \frac{1}{2} - \lambda & \frac{2-i}{2} & \frac{1-i}{2} \\ \frac{3}{2} - s & \frac{3}{2} - t & \frac{1}{2} + s + t - i \end{matrix} \middle| 1 \right] N_i^{st}(m, n), \quad (\text{A.8})$$

$q = \frac{1}{4}$ and:

$$N_i^{st}(m, n) = \sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} (s-1+m+1)_{k-i+1} (s-1-m+1)_{-k} (t-1+n+1)_{-k} (t-1-n+1)_{k-i+1}. \quad (\text{A.9})$$

Where the $(n)_k$ are the ascending Pochhammer symbols. The lone star product contains also may contain the generator V_0^1 which does not make part of the algebra, it appears symmetrically in the products, so we can use it to define the trace, the trace of any product will be precisely the piece accompanying this generator. using the lone star product we get:

$$\text{tr}(V_{m_s}^s V_{-m_s}^s) \equiv \frac{6}{1-\lambda^2} \frac{(-1)^{m_s} 2^{3-2s} \Gamma(s+m_s) \Gamma(s-m_s)}{(2s-1)!!(2s-3)!!} \prod_{\sigma=1}^{s-1} (\lambda^2 - \sigma^2) \quad (\text{A.10})$$

Notice that the trace immediately factors out the ideal when we truncate to $\lambda = N$. The $\mathfrak{hs}(\lambda)$ can be thought as constructed from $\mathfrak{sl}(2, \mathbb{R})$, and any element can be formally written as:

$$V_m^s = (-1)^{s+m-1} \frac{(s+m-1)!}{(2s-2)!} \underbrace{[L_{-1}, \dots, [L_{-1}, [L_{-1}, L_1^{s-1}]]}_{s-1-m} \quad (\text{A.11})$$

where $V_1^2 = L_1$. When we truncate the algebra to $\lambda = 3$, we get $\mathfrak{sl}(3, \mathbb{R})$, and we have two different embeddings. The Killing metric in the principal embedding for the ordering given in (3.15)

$$g_{ab} = \text{tr}(V_a V_b) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{A.12})$$

The Killing metric in diagonal embedding for the ordering given in (3.61)

$$g_{ab} = \text{tr}(V_a V_b) = \begin{pmatrix} -\frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 \end{pmatrix} \quad (\text{A.13})$$

The differences respect to killing metric presented in the (A.1), corresponds just to a different normalization of the trace and ins not relevant at all.

B. USEFUL RESULTS OF

Here we report some results that were useful during the computations in section (3.1.1). In particular, the solution to the conditions

$$\begin{aligned} (\delta \mathcal{L}_1^{(0)})_{\delta \rightarrow (\delta_\Lambda)|_{\mu_3 \rightarrow 0}} &= (\delta_\Lambda \mathcal{L}) \Big|_{\text{At } \mu_3 \text{ \& } x_2 \rightarrow 0}, \\ (\delta \mathcal{W}_1^{(0)})_{\delta \rightarrow (\delta_\Lambda)|_{\mu_3 \rightarrow 0}} &= (\delta_\Lambda \mathcal{W}) \Big|_{\text{At } \mu_3 \text{ \& } x_2 \rightarrow 0}, \end{aligned} \quad (\text{B.1})$$

where we remind the reader that by $(\delta \dots)|_{\delta \rightarrow \delta_\Lambda}$ we mean:

- Take the functional differential of ... in terms of $(\delta \mathcal{L}^{(0)}, \delta \mathcal{W}^{(0)})$ and thereafter substitute δ by δ_Λ . The expressions for $(\delta_\Lambda \mathcal{L}^{(0)}, \delta_\Lambda \mathcal{W}^{(0)})$ are reported in (3.36). The expressions for $(\delta_\Lambda \mathcal{L}, \delta_\Lambda \mathcal{W})$ are reported in (3.29).

The most general solution to (B.1) read out

$$\begin{aligned} \mathcal{L}_1^{(0)} &= 3c_1 \mathcal{W}^{(0)} + c_2 \partial_1 \mathcal{L}^{(0)} + 2c_1 x_1 \partial_1 \mathcal{W}^{(0)} \\ \mathcal{W}_1^{(0)} &= -c_1 \left(\frac{8}{3} \mathcal{L}^{(0)2} + \frac{3}{4} \partial_1^2 \mathcal{L}^{(0)} \right) + c_2 \partial_1 \mathcal{W}^{(0)} - c_1 x_1 \left(\frac{8}{3} \partial_1 \mathcal{L}^{(0)} + \frac{1}{6} \partial_1^3 \mathcal{L}^{(0)} \right) \\ \epsilon_1^{(0)} &= -c_1 \left(\frac{8}{3} \eta^{(0)} \mathcal{L}^{(0)} + \frac{1}{4} \partial_1^2 \eta^{(0)} \right) + c_2 \partial_1 \epsilon^{(0)} + c_1 x_1 \left(\frac{8}{3} \partial_1 \eta^{(0)} \mathcal{L}^{(0)} + \frac{1}{6} \partial_1^3 \eta^{(0)} \right) \\ \eta_1^{(0)} &= c_1 \epsilon^{(0)} + c_2 \partial_1 \eta^{(0)} - 2c_1 x_1 \partial_1 \epsilon^{(0)}. \end{aligned} \quad (\text{B.2})$$

It is straightforward to check that (B.2) coincides with (3.38) for $c_1 = 1$ and $c_2 = 0$. In fact this is the unique choice out of (B.2) that allows to integrate the differential of charge to (3.39).

It is also useful to write down the most general choice of $(\mathcal{L}_1^{(0)}, \mathcal{W}_1^{(0)}, \epsilon_1^{(0)}, \eta_1^{(0)})$ that is consistent without explicit dependence on ϕ and dimensional analysis. It is given by

$$\begin{aligned} \mathcal{L}^{(0)}_{1hom} &= c_3 \mathcal{W} + c_4 \partial_1 \mathcal{L}, & \mathcal{W}^{(0)}_{1hom} &= c_5 \mathcal{L}^2 + c_6 \partial_1^2 \mathcal{L} + c_7 \partial_1 \mathcal{W}, \\ \epsilon_1^{(0)}_{hom} &= c_8 \partial_1 \epsilon + c_9 \partial_1^2 \eta + 2c_{10} \mathcal{L} \eta, & \eta_1^{(0)}_{hom} &= c_{11} \epsilon + c_{12} \partial_1 \eta. \end{aligned} \quad (\text{B.3})$$

We use (B.3) to show that (3.24) is not isomorphic to \mathcal{W}_3 .

C. MISCELLANEOUS FOR $\mathfrak{hs}(\lambda) \oplus \mathfrak{hs}(\lambda)$ SOLUTIONS

C.1 Unicity of the Choice $\nu_0 = \frac{1}{2}$, $\nu_{i>0} = 0$ for $0 < \lambda < 1$

\prod_4^5 Here we show how the only solution to the integrability condition (4.21) in the region $0 < \lambda < 1$ is the trivial one $n_0 = 1$. First we write down the first 4×4 block of the upper triangular matrix M

$$\begin{pmatrix} 1 & \frac{4(\lambda^2-4)}{15} & \frac{4(\lambda^2-4)(11\lambda^2-71)}{315} & \frac{4(\lambda^2-4)(107\lambda^4-1630\lambda^2+6563)}{4725} \\ 0 & \frac{12 \prod_{\sigma=2}^3 \sqrt{(\lambda^2-\sigma^2)}}{5\sqrt{14}} & \frac{4(7\lambda^2-67) \prod_{\sigma=2}^3 \sqrt{(\lambda^2-\sigma^2)}}{15\sqrt{14}} & \frac{4 \prod_{\sigma=2}^3 \sqrt{(\lambda^2-\sigma^2)}(893\lambda^4-19090\lambda^2+113957)}{2475\sqrt{14}} \\ 0 & 0 & \frac{8\sqrt{\frac{5}{11}} \prod_{\sigma=2}^5 \sqrt{(\lambda^2-\sigma^2)}}{21} & \frac{80\sqrt{\frac{5}{11}} \prod_{\sigma=2}^5 \sqrt{(\lambda^2-\sigma^2)}(5\lambda^2-89)}{819} \\ 0 & 0 & 0 & \frac{32\sqrt{\frac{7}{5}} \prod_{\sigma=2}^7 \sqrt{(\lambda^2-\sigma^2)}}{429} \end{pmatrix}. \quad (\text{C.1})$$

The eigenvalues can be checked to be greater or equal than one in $0 < \lambda < 1$. In fact they grow as the diagonal index i grows. Next we show this excludes the presence of any other solution. Be the following definition and couple of facts

$$n_O^i \equiv O_j^i n^j, \quad OM^TMO^T = \text{Diag}((M^{ii})^2), \quad O^TO = 1. \quad (\text{C.2})$$

As $(M^{ii})^2 \geq 1$ it is clear that

$$\sum_{i=1}^{\infty} \left((M \cdot n)^i \right)^2 = \sum_{i=1}^{\infty} (M^{ii})^2 (n_{O_i})^2 \geq \sum_{i=1}^{\infty} n_{O_i}^2 = \sum_{i=1}^{\infty} n_i^2 \geq 1. \quad (\text{C.3})$$

The saturation in (C.3) comes when one of the integers n_i is one. As $(M^{ii})^2 = 1$ only if $i = 1$ thence the only solution to (4.21) is the trivial one. Notice however that our conclusions do breakdown when we are out of the region $0 < \lambda < 1$. This is, to define a new solution we just need to tune up λ in such a way that for a given i , $M^{ii} = \pm 1$.

C.2 Solutions with $z < 1$

Here we study the fluctuations for an specific background $z < 1$. We take as a toy example the case $\bar{\mu}_3 = -\mu_3 \neq 0$. The secular polynomial reads out

$$ik = ik'_r - 2\mu_3 \left(\omega^2 + k_r'^2 + \frac{\lambda^2 - 1}{3} \right), \quad (\text{C.4})$$

whose roots are

$$k'_{\pm} = \frac{-i + \sqrt{-1 + 8ik\mu_3 - \frac{16}{3}(\lambda^2 + 3\omega^2 - 1)\mu_3^2}}{4\mu_3}. \quad (\text{C.5})$$

From the quantization condition (4.75)

$$\begin{aligned} w_{1-n}^{\pm} &= -i\frac{1}{2}(1 + 2n + \lambda) + \delta_{1\ z<1}^{\pm}, \\ w_{2-n}^{\pm} &= -i\frac{1}{2}(1 + 2n + \lambda) + \delta_{2\ z<1}^{\pm}, \end{aligned} \quad (\text{C.6})$$

where the \pm refer to the \pm in (C.5) and the $(1, 2)$ refer to the $(+, -)$ in (4.75) respectively, and

$$\begin{aligned} \delta_{1\ z<1}^{\pm} &= \frac{3i \mp \sqrt{-1 + 8(-1 + 2ik - 2n - \lambda)\mu_3 + \frac{16}{3}(5 + 12n^2 + 6\lambda + \lambda^2 + 12n(1 + \lambda))\mu_3^2}}{8\mu_3}, \\ \delta_{2\ z<1}^{\pm} &= \frac{-3i \pm \sqrt{-1 + 8(1 + 2ik + 2n + \lambda)\mu_3 + \frac{16}{3}(5 + 12n^2 + 6\lambda + \lambda^2 + 12n(1 + \lambda))\mu_3^2}}{8\mu_3}. \end{aligned} \quad (\text{C.7})$$

We can also study the case $\bar{\mu}_3 = \mu_3$, we get in this case from (4.70):

$$k' = \frac{k + 4ik\omega\mu_3}{1 + 16\omega^2\mu_3^2}. \quad (\text{C.8})$$

We get just one root, which means that after the folding process of section (4.2), the final equation obtained is of second order, as can be explicitly checked. The QNM in this case are given by:

$$\begin{aligned} \omega_{1\pm} &= \frac{-i - 4i(1 + 2n + \lambda)\mu_3 \mp \sqrt{-1 + 8(1 - 2ik + 2n + \lambda)\mu_3 - 16(1 + 2n + \lambda)^2\mu_3^2}}{8\mu_3}, \\ \omega_{2\pm} &= \frac{-i - 4i(1 + 2n + \lambda)\mu_3 \mp \sqrt{-1 + 8(1 + 2ik + 2n + \lambda)\mu_3 - 16(1 + 2n + \lambda)^2\mu_3^2}}{8\mu_3}. \end{aligned} \quad (\text{C.9})$$

In section (4) we have given the metric for these solutions (4.22). Propagation in Lifshitz metrics with $z < 1$ is typically associated with the presence of superluminal excitations in the dual field theory, see for instance [111, 112]. For each one of our blocks r we can make use of the AdS/CFT dictionary. The dispersion relations for the corresponding physical excitation, n , is given by the condition for a pole in the retarded 2-point function (4.75) and the expression for the auxiliary momentum k'_r of the given block in terms of k and w are given in (C.5) and (C.8) respectively. The wavefront velocity $v_f = \lim_{\omega \rightarrow \infty} \frac{\omega}{k_R(\omega, n)}$, [113], can be computed to be

$$v_{f1} = \lim_{\omega \rightarrow \infty} \frac{\omega}{-\omega + 4\omega\mu_3 + 8n\omega\mu_3 + 4\lambda\omega\mu_3} = \frac{1}{-1 + 4\mu_3 + 8n\mu_3 + 4\lambda\mu_3}, \quad (\text{C.10})$$

$$v_{f2} = \lim_{\omega \rightarrow \infty} \frac{\omega}{\omega + 4\omega\mu_3 + 8n\omega\mu_3 + 4\lambda\omega\mu_3} = \frac{1}{1 + 4\mu_3 + 8n\mu_3 + 4\lambda\mu_3}. \quad (\text{C.11})$$

We end up by noticing that for $|\mu_3| \geq \frac{1}{2(1+\lambda)}$ there are no superluminal modes ($|v_f| \leq 1$) in these examples. But for other values there is a finite number of them. However the tale of large n excitations have all $|v_f| \leq 1$.

C.3 C_{BTZ} and Differential operators

Finally we present some differential operators that were referenced in the main body of the text. The Klein Gordon operator in ρ coordinates:

$$D_2 \equiv \frac{d^2}{d\rho^2} + \frac{2(e^{4\rho} + 1)}{(e^{4\rho} - 1)} \frac{d}{d\rho} + \frac{(1 - \lambda^2)(e^{8\rho} - 1)}{(e^{4\rho} - 1)^2} - \frac{2(2(k^2 - \omega^2)(e^{2\rho} + e^{6\rho}) + \lambda^2 - 1 - e^{4\rho}(4k^2 + 4\omega^2 + \lambda^2 - 1))}{(e^{4\rho} - 1)^2}. \quad (C.12)$$

The operator D_4 for the background $\mu_3 \neq 0$

$$D_4(z) \equiv \partial_z^4 - \frac{2iw(z-1)+2(\lambda-4)z+4}{(z-1)z} \partial_z^3 + \left(\frac{-3(z-1)z+6i\mu_3(z-1)z(k+2w)}{12\mu_3^2(z-1)^2z^2} - \frac{3w^2(z-1)^2-9iw(z-1)((\lambda-3)z+1)+z((\lambda-18)\lambda-(\lambda-4)(4\lambda-11)z+44)-6}{3(z-1)^2z^2} \right) \partial_z^2 + \frac{(w(z-1)-i((\lambda-2)z+1))(6k\mu_3+4\mu_3(3w+(\lambda-2)\mu_3(3w-i(\lambda-4))))+3i}{12\mu_3^2(z-1)^2z^2} \partial_z - \frac{(-i(\lambda-1)(2(\lambda-2)\mu_3+3)+3k+3w)(-i(\lambda-1)(2(\lambda-2)\mu_3-3)+3k+12i\mu_3w^2+3w(4(\lambda-1)\mu_3-1))}{144\mu_3^2(z-1)^2z^2}. \quad (C.13)$$

The differential operator $\overset{(1)}{D}_{GK}$ that we make reference to in section (4.4)

$$\overset{(1)}{D}_{GK} \equiv \frac{64ie^{2\rho}(3e^{2\rho} - 1)k}{(e^{2\rho} - 1)^2(1 + e^{2\rho})^3(\lambda^2 - 1)} \frac{d}{d\rho} + \frac{8k \left(\frac{1-11k^2-\omega^2-\lambda^2+e^{6\rho}(-7k^2+3\omega^2-5\lambda^2-11)}{(e^{2\rho}-1)^3} + e^{4\rho}(3k^2 + 9\omega^2 + \lambda^2 - 1) \right)}{-ie^{-2\rho}(1 + e^{2\rho})^4(\lambda^2 - 1)} + \frac{8k \left(\frac{e^{8\rho}(42\omega^2+6k^2+2\lambda^2-2)+e^{4\rho}(29-15k^2+59\omega^2+3\lambda^2)+e^{2\rho}(27k^2+25\omega^2+\lambda^2-17)}{(e^{2\rho}-1)^3} \right)}{-ie^{-2\rho}(1 + e^{2\rho})^4(\lambda^2 - 1)}. \quad (C.14)$$

Finally, we give the master field C for the BTZ Background up to spin 4. We have used the Fourier basis (4.53) and redefined $C_0^1 \equiv C$:

$$C_{\pm 1}^2 = \frac{6ie^\rho (\mp(e^{2\rho} - 1)k + (e^{2\rho} + 1)\omega) C[\rho]}{(e^{2\rho} - 1)(e^{2\rho} + 1)(\lambda^2 - 1)}, \quad (\text{C.15})$$

$$C_0^2 = -\frac{6C'[\rho]}{\lambda^2 - 1}, \quad (\text{C.16})$$

$$C_0^3 = \frac{30 \left(\frac{6(k^2 - \omega^2)(e^{2\rho} + e^{6\rho})}{\lambda^2 - 1} + 1 + e^{8\rho} - 2e^{4\rho} \left(\frac{6k^2 + 6\omega^2}{\lambda^2 - 1} + 1 \right) \right) C[\rho]}{(e^{4\rho} - 1)^2(\lambda^2 - 4)} - \frac{90(e^{8\rho} - 1)C'[\rho]}{(e^{4\rho} - 1)^2(4 - 5\lambda^2 + \lambda^4)}, \quad (\text{C.17})$$

$$C_{\pm 1}^3 = \frac{\left(\frac{\mp(e^{3\rho} - e^\rho)}{(1 + e^{2\rho})^2} k + \omega \frac{(e^{3\rho} + e^\rho)}{(e^{2\rho} - 1)^2} \right) C[\rho] + \left(\frac{\pm e^\rho}{(1 + e^{2\rho})} k - \frac{e^\rho}{(e^{2\rho} - 1)} \omega \right) C'[\rho]}{\frac{(4 - 5\lambda^2 + \lambda^4)}{60i}}, \quad (\text{C.18})$$

$$C_{\pm 2}^3 = -\frac{30 \left(\frac{\mp e^\rho}{(e^{2\rho} + 1)} k + \frac{e^\rho}{(e^{2\rho} - 1)} \omega \right)^2 C[\rho] + \frac{30e^{2\rho}}{(e^{4\rho} - 1)} C'[\rho]}{(4 - 5\lambda^2 + \lambda^4)} \quad (\text{C.19})$$

$$C_0^4 = \frac{\left((e^{2\rho} + 4e^{6\rho} + e^{10\rho}) \frac{(k^2 - \omega^2)}{\lambda^2 - 1} + \left(\frac{1 + e^{12\rho}}{8} - (e^{4\rho} + e^{8\rho}) \left(\frac{3k^2 + 3\omega^2}{\lambda^2 - 1} + \frac{1}{8} \right) \right) \right) C[\rho]}{\frac{(e^{4\rho} - 1)^3(\lambda^2 - 9)(\lambda^2 - 4)}{5600}} - \frac{\left((e^{2\rho} + e^{6\rho})(k^2 - \omega^2) + \frac{(1 + e^{8\rho})(11 + \lambda^2)}{10} - 2e^{4\rho}(k^2 + \omega^2 + \frac{\lambda^2 - 29}{10}) \right) C'[\rho]}{\frac{(e^{4\rho} - 1)^2(\lambda^2 - 9)(\lambda^2 - 4)(\lambda^2 - 1)}{42000}}, \quad (\text{C.20})$$

$$C_{\pm 1}^4 = \left(\frac{\pm k \left(\frac{(1 + \lambda^2)(1 + e^{8\rho})}{5} - (e^{2\rho} + e^{6\rho})(2 + \omega^2) - 2e^{4\rho}(\omega^2 + \frac{\lambda^2 - 9}{5}) \right)}{\frac{ie^{-\rho}(e^{2\rho} - 1)^2(e^{2\rho} + 1)^3(\lambda^2 - 9)(\lambda^2 - 4)(\lambda^2 - 1)}{2100}} + \frac{\pm e^{2\rho}k^3 - e^{2\rho} \frac{(e^{2\rho} + 1)}{(e^{2\rho} - 1)} k^2 \omega - \frac{(e^{2\rho} + 1)^3}{(e^{2\rho} - 1)^3} \omega \left(\frac{(1 + \lambda^2)(1 + e^{4\rho})}{5} + \frac{e^{2\rho}(8 - 5\omega^2 - 2\lambda^2)}{5} \right)}{\frac{ie^{-\rho}(e^{2\rho} + 1)^3(\lambda^2 - 9)(\lambda^2 - 4)(\lambda^2 - 1)}{2100}} \right) C[\rho] - \frac{2(e^{2\rho} - 1) \left(\pm(e^{2\rho} - e^{4\rho} + \frac{e^{6\rho} - 1}{2})k - (e^{2\rho} + e^{4\rho} + \frac{e^{6\rho} + 1}{2})\omega \right) C'[\rho]}{\frac{ie^{-\rho}(e^{2\rho} + 1)^2(\lambda^2 - 9)(\lambda^2 - 4)(\lambda^2 - 1)}{2100}}, \quad (\text{C.21})$$

$$C_{\pm 2}^4 = -420e^{2\rho} \left(\frac{\pm 8k\omega + (1 - \lambda^2 \mp 4k\omega + 4\omega^2)(1 + e^{8\rho}) + 2e^{4\rho}(1 + 20\omega^2)}{(e^{4\rho} - 1)^3(\lambda^2 - 9)(\lambda^2 - 4)(\lambda^2 - 1)} + \frac{(20e^{4\rho} - 12(e^{2\rho} + e^{6\rho}) + 2(1 + e^{8\rho}))(k^2 - \omega^2)}{(e^{4\rho} - 1)^3(\lambda^2 - 9)(\lambda^2 - 4)(\lambda^2 - 1)} \right) C[\rho] + 420e^{2\rho} \frac{(\pm 4k\omega - 2e^{2\rho}(k^2 - \omega^2) + (1 + e^{4\rho})(k^2 \mp 2k\omega + \omega^2 - 4)) C'[\rho]}{(e^{4\rho} - 1)^2(\lambda^2 - 9)(\lambda^2 - 4)(\lambda^2 - 1)}, \quad (\text{C.22})$$

$$C_{\pm 3}^4 = \frac{\left(\frac{\pm k(3\omega^2 + e^{4\rho}(3\omega^2 - 2) + e^{2\rho}(4 + 6\omega^2) - 2)}{(e^{2\rho} - 1)^2} \pm k^3 - \frac{3(1 + e^{2\rho})k^2\omega}{e^{2\rho} - 1} - \frac{(1 + e^{2\rho})^3\omega(\omega^2 - 2)}{(e^{2\rho} - 1)^3} \right) C[\rho]}{\frac{-ie^{-3\rho}(e^{2\rho} + 1)^3(\lambda^2 - 9)(\lambda^2 - 4)(\lambda^2 - 1)}{140}} + \frac{(\pm(e^{2\rho} - 1)k - (1 + e^{2\rho})\omega) C'[\rho]}{\frac{-ie^{3\rho}(e^{4\rho} - 1)^3(\lambda^2 - 9)(\lambda^2 - 4)(\lambda^2 - 1)}{420}}. \quad (\text{C.23})$$

The primes stand for derivative along ρ , and one can recover the result in coordinate space (t, ϕ) by replacing $k \rightarrow -i\partial_\phi$ and $\omega \rightarrow -i\partial_t$. Notice that all these higher spin components are generically singular at the horizon.

D. CONVENTIONS FOR $\mathcal{N} = 2$ THEORIES IN 4D

We are using ‘flat’ γ -matrices:

$$\gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \bar{\sigma}_a & 0 \end{pmatrix}, \quad (\text{D.1})$$

with

$$\sigma_a = -i\tau_a \quad \bar{\sigma}_a = i\tau_a, a = 1, 2, 3 \quad \text{and} \quad \sigma_4 = \bar{\sigma}_4 = \mathbb{1}_{2 \times 2}$$

And the τ ’s are the standard Pauli matrices, with that we have: $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$ with η_{ab} the frame metric $\eta = \mathbb{K}_{4 \times 4}$.

$$\gamma_{ab} = \frac{1}{2}(\gamma_a \gamma_b - \gamma_b \gamma_a) = \begin{pmatrix} \frac{(\sigma_a \bar{\sigma}_b - \sigma_b \bar{\sigma}_a)}{2} & 0 \\ 0 & \frac{(\bar{\sigma}_a \sigma_b - \bar{\sigma}_b \sigma_a)}{2} \end{pmatrix} = \begin{pmatrix} \sigma_{ab} & 0 \\ 0 & \bar{\sigma}_{ab} \end{pmatrix}, \quad (\text{D.2})$$

σ_{ab} and $\bar{\sigma}_{ab}$ are self-dual and anti self-dual respectively.

we work with four component spinor $\Xi_A = \begin{pmatrix} \xi_{\alpha A} \\ \tilde{\xi}_{\dot{\alpha} A} \end{pmatrix}$. Spinors $\xi_{\alpha A}$ and $\tilde{\xi}_{\dot{\alpha} A}$ transform as doublets under two independent $SU(2)$ subgroups of the 4D rotation group, these are the dotted and undotted Greek indices. While they transform also as doublets of the $SU(2)_R$. All these indices are rise and lower with the ϵ invariant tensor of $SU(2)$:

$$(\epsilon^{\alpha\beta}) = (\epsilon_{\dot{\alpha}\dot{\beta}}) = (\epsilon^{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{with} \quad \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon^{\dot{\beta}\dot{\mu}} = \delta_{\dot{\alpha}}^{\dot{\mu}}, \quad \text{and} \quad \epsilon^{\mu\beta} \epsilon_{\beta\alpha} = \delta_{\alpha}^{\mu} \quad (\text{D.3})$$

The conventions for contracting, lowering and raising the spinor indices are very much like in [114]. In four component notation we use:

$$\mathcal{C} = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix} \quad (\text{D.4})$$

The covariant and Lie derivative act on the spinors as follows:

$$\begin{aligned} \nabla_m \Xi_A &= \partial_m \Xi_A + \frac{1}{4} \omega_m^{ab} \gamma_{ab} \Xi_A, \\ \mathcal{L}_K \Xi_A &= K^m \nabla_m \Xi_A + \frac{1}{4} (\nabla_n K_m) \gamma^{nm} \Xi_A, \end{aligned} \quad (\text{D.5})$$

E. MANIPULATIONS FOR THE 1-LOOP DETERMINANT

We present some algebraic manipulations used for the 1-loop determinant

$$(\mathcal{V} + \mathcal{V}_{GF})|_{Quad} = (\hat{\mathcal{Q}}\mathbf{X}, \Xi) \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \hat{\mathcal{Q}}\Xi \end{pmatrix}, \quad (\text{E.1})$$

Then

$$\hat{\mathcal{Q}}(\mathcal{V} + \mathcal{V}_{GF})|_{Quad} = XK_{Bosons}X + \Xi K_{Fermions}\Xi, \quad (\text{E.2})$$

Considering $\hat{\mathcal{Q}}^2(\mathcal{V} + \mathcal{V}_{GF})|_{Quad}$.

$$\begin{aligned} \hat{\mathcal{Q}}(\mathcal{V} + \mathcal{V}_{GF})|_{Quad} = & \begin{pmatrix} X & \hat{\mathcal{Q}}\Xi \end{pmatrix} \begin{pmatrix} -\hat{\mathcal{Q}}^2 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{D} \begin{pmatrix} X \\ \hat{\mathcal{Q}}\Xi \end{pmatrix} - \\ & \begin{pmatrix} \hat{\mathcal{Q}}X & \Xi \end{pmatrix} \mathbb{D} \begin{pmatrix} 1 & 0 \\ 0 & \hat{\mathcal{Q}}^2 \end{pmatrix} \begin{pmatrix} \hat{\mathcal{Q}}X \\ \Xi \end{pmatrix}, \end{aligned} \quad (\text{E.3})$$

where $\mathbb{D} \equiv \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}$.

$$\begin{aligned} K_{Bosons} &= \begin{pmatrix} -\hat{\mathcal{Q}}^2 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{D}, \\ K_{Fermions} &= -\mathbb{D} \begin{pmatrix} 1 & 0 \\ 0 & \hat{\mathcal{Q}}^2 \end{pmatrix}, \end{aligned} \quad (\text{E.4})$$

Since $\hat{Q}^2(\mathcal{V} + \mathcal{V}_{GF}) = 0$ we will have $[\hat{Q}^2, D_{ij}] = 0$, as can be readily seen

$$\begin{aligned}
& \hat{Q}^2(\mathcal{V} + \mathcal{V}_{GF})_{Quad} = \\
& (\hat{Q}\mathbf{X} \quad \hat{Q}^2\Xi) \begin{pmatrix} -\hat{Q}^2 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{D} \begin{pmatrix} \mathbf{X} \\ \hat{Q}\Xi \end{pmatrix} + (\mathbf{X} \quad \hat{Q}\Xi) \begin{pmatrix} -\hat{Q}^2 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{D} \begin{pmatrix} \hat{Q}\mathbf{X} \\ \hat{Q}^2\Xi \end{pmatrix} \\
& - (\hat{Q}^2\mathbf{X} \quad \hat{Q}\Xi) \mathbb{D} \begin{pmatrix} 1 & 0 \\ 0 & -\hat{Q}^2 \end{pmatrix} \begin{pmatrix} \hat{Q}\mathbf{X} \\ \Xi \end{pmatrix} + (\hat{Q}\mathbf{X} \quad \Xi) \mathbb{D} \begin{pmatrix} 1 & 0 \\ 0 & \hat{Q}^2 \end{pmatrix} \begin{pmatrix} \hat{Q}^2\mathbf{X} \\ \hat{Q}\Xi \end{pmatrix} \\
& = (cc\hat{Q}\mathbf{X} \quad \Xi) \begin{pmatrix} 1 & 0 \\ 0 & -\hat{Q}^2 \end{pmatrix} \begin{pmatrix} -\hat{Q}^2 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{D} \begin{pmatrix} c\mathbf{X} \\ \hat{Q}\Xi \end{pmatrix} + (\mathbf{X} \quad \hat{Q}\Xi) \begin{pmatrix} -\hat{Q}^2 & 0 \\ 0 & 1 \end{pmatrix} \\
& \mathbb{D} \begin{pmatrix} 1 & 0 \\ 0 & \hat{Q}^2 \end{pmatrix} \begin{pmatrix} \hat{Q}\mathbf{X} \\ \Xi \end{pmatrix} - (\mathbf{X} \quad \hat{Q}\Xi) \begin{pmatrix} -\hat{Q}^2 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{D} \begin{pmatrix} 1 & 0 \\ 0 & \hat{Q}^2 \end{pmatrix} \begin{pmatrix} \hat{Q}\mathbf{X} \\ \Xi \end{pmatrix} \\
& + (\hat{Q}\mathbf{X} \quad \Xi) \mathbb{D} \begin{pmatrix} 1 & 0 \\ 0 & \hat{Q}^2 \end{pmatrix} \begin{pmatrix} \hat{Q}^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \hat{Q}\Xi \end{pmatrix}, \\
& = (\hat{Q}\mathbf{X} \quad \Xi) \begin{pmatrix} -\hat{Q}^2 & 0 \\ 0 & -\hat{Q}^2 \end{pmatrix} \mathbb{D} \begin{pmatrix} \mathbf{X} \\ \hat{Q}\Xi \end{pmatrix} + (\hat{Q}\mathbf{X} \quad \Xi) \mathbb{D} \begin{pmatrix} \hat{Q}^2 & 0 \\ 0 & \hat{Q}^2 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \hat{Q}\Xi \end{pmatrix}.
\end{aligned} \tag{E.5}$$

It is clear then that the last equation vanish only if $[\hat{Q}^2, D_{ij}] = 0$. Then using (E.4) we have that:

$$\begin{pmatrix} 1 & 0 \\ 0 & -\hat{Q}^2 \end{pmatrix} K_{Bosons} = K_{Fermions} \begin{pmatrix} \hat{Q}^2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{E.6}$$

And therefore we get (up to signs ambiguities) that

$$Z_{1-loop} = \left(\frac{\det K_{Fermions}}{\det K_{Bosons}} \right)^{1/2} = \left(\frac{\det_{\text{coKer} D_{10}} \hat{Q}^2}{\det_{\text{Ker} D_{10}} \hat{Q}^2} \right)^{1/2}, \tag{E.7}$$

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